Pseudozero Set of Multivariate Polynomials

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Abstract

The pseudozero set of a system P of polynomials in n variables is the subset of $\mathbf{C}^n$ consisting of the union of the zeros of all polynomial systems Q that are near to P in a suitable sense. This concept arises naturally in Scientific Computing where data often have a limited accuracy. When the system is defined by polynomials with complex coefficients, the pseudozero set has already been studied. In this poster, we focus on polynomial systems where both the coefficients of the polynomial system and of the perturbed one are real. We provide an explicit formula to compute this pseudozero set. At last, we analyze different methods to visualize this set.

Introduction

A *monomial* in the n variables $z_1, \dots, z_n$ is the power product

$$z^j := z_1^{j_1} \cdots z_n^{j_n}, \quad \text{with } j = (j_1, \dots, j_n) \in \mathbf{N}^n;$$

j is the *exponent* and $|j| := \sum_{\sigma=1}^n j_\sigma$ the *degree* of the monomial z^j.

Definition 1. A *complex (real) polynomial* in n variables is a finite linear combination of monomials in n variables with coefficients from $\mathbf{C}$ (from $\mathbf{R}$),

$$p(z) = p(z_1, \dots, z_n) = \sum_{(j_1,\dots,j_n)\in J}^{n} a_{j_1 \cdots j_n} z_1^{j_1} \cdots z_n^{j_n} = \sum_{j \in J} a_j z^j.$$

$\mathcal{P}^n(\mathbf{C})$ ($\mathcal{P}^n(\mathbf{R})$) represents the set of all complex (real) polynomials in n variables. We collect the coefficients of a polynomial into a vector $a = (\dots, a_j, \dots, j \in J)^T$ and its monomials into a vector $\mathbf{z} = (\dots, z^j, \dots, j \in J)^T$.

Let $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{K})$ with $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ be a polynomial in n variables. We denote by $|J|$ the number of elements of J. If $|J| = M$ and let $\|\cdot\|$ be a norm on $\mathbf{K}^M$, $\|p\|$ is the norm of the vector $a = (\dots, a_j, \dots, j \in J)$.

Given $\varepsilon > 0$, the ε-neighborhood $N_\varepsilon(p)$ of the polynomial $p \in \mathcal{P}^n(\mathbf{K})$ is the set of all polynomials of $\mathcal{P}^n(\mathbf{K})$, close enough to p, that is to say, the set of polynomials $\widetilde{p} = \sum_{j\in\widetilde{J}} \widetilde{a}_j z^j \in \mathcal{P}^n(\mathbf{K})$ with support $\widetilde{J} \subset J$ and $\|\widetilde{p} - p\| \le \varepsilon$.

Given a norm $\|\cdot\|$ on $\mathbf{K}^N$ with $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, the *dual norm* is defined by $\|x\|_* := \sup_{\|y\|=1} |y^T x|$. Given a vector $x \in \mathbf{K}^N$, there exists a *dual vector* $y \in \mathbf{K}^N$ with $\|y\| = 1$ satisfying $x^T y = \|x\|_*$.

Definition 2. A value $z \in \mathbf{K}^n$ is a ε-pseudozero of a polynomial $p \in \mathcal{P}^n$ if it is a zero of some polynomial $\widetilde{p}$ in $N_\varepsilon(p)$.

Definition 3. The ε-pseudozero set of a polynomial $p \in \mathcal{P}^n$ (denoted by $Z_\varepsilon(p)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(p) := \{z \in \mathbf{K}^n : \exists \widetilde{p} \in N_\varepsilon(p), \quad \widetilde{p}(z) = 0\}.$$

Given $\varepsilon > 0$ and a *system* of polynomials $P = \{p_1, \dots, p_k\}, \quad k \in \mathbf{N}$, the ε-neighborhood $N_\varepsilon(P)$ is the set of systems of polynomials $\widetilde{P} = \{\widetilde{p}_1, \dots, \widetilde{p}_k\}$ close enough to P, that is with $\widetilde{p}_j \in N_\varepsilon(p_j)$ for $j = 1, \dots, k$.

Definition 4. A value $z \in \mathbf{K}^n$ is a *ε-pseudozero* of a polynomial system P if it is a zero of a system of polynomials $\widetilde{P}$ in $N_\varepsilon(P)$.

Definition 5. The *ε-pseudozero set* of a system of polynomials P (denoted by $Z_\varepsilon(P)$) is the set of all the ε-pseudozeros,

$$Z_\varepsilon(P) := \{z \in \mathbf{K}^n : \exists \widetilde{P} \in N_\varepsilon(P), \quad \widetilde{P}(z) = 0\}.$$

Pseudozero set of complex multivariate polynomials

Theorem 1 below provides a computable counterpart of the pseudozero set.

Theorem 1 (Stetter [5]). *The complex ε-pseudozero set of $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{C})$ verifies*

$$Z_\varepsilon(p) = \left\{ z \in \mathbf{C}^n : g(z) := \frac{|p(z)|}{\|\mathbf{z}\|_*} \le \varepsilon \right\}$$

where $\mathbf{z} := (\dots, |z|^j, \dots, j \in J)^T$.

This theorem can be immediately extended to systems of polynomials.

Corollary 2 (Stetter [4]). *The complex ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbf{N}$ verifies*

$$Z_\varepsilon(P) = \left\{ z \in \mathbf{C}^n : \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \le \varepsilon \text{ for } l = 1, \dots, k \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$.

For the next theorem, we restrict our attention to situations where P as well as all the systems in $N_\varepsilon(P)$ are 0-dimensional, that is, if the solution of the system is non-empty and finite.

Theorem 3 (Stetter [4]). *Under the above assumptions, each system $\widetilde{P} \in N_\varepsilon(P)$ has the same number of zeros (counting multiplicities) in a fixed pseudozero set connected component of $Z_\varepsilon(P)$.*

Pseudozero set of real multivariate polynomials

A real ε-neighborhood of p is the set

$$N_\varepsilon^R(p) = \{\widetilde{p} \in \mathcal{P}^n(\mathbf{R}) : \|p - \widetilde{p}\| \le \varepsilon\}.$$

Then the real ε-pseudozero set of p is defined by

$$Z_\varepsilon^R(p) = \left\{ z \in \mathbf{C}^n : \widetilde{p}(z) = 0 \text{ for } \widetilde{p} \in N_\varepsilon^R(p) \right\}.$$

$Z(p)$ is the set of the roots of p.

Following Theorem 4 provides a computable counterpart of this definition. We define for $x, y \in \mathbf{R}^N$,

$$d(x, \mathbf{R}y) = \inf_{\alpha\in\mathbf{R}} \|x - \alpha y\|_*,$$

the distance of a point x from the linear subspace $\mathbf{R}y = \{\alpha y, \alpha \in \mathbf{R}\}$.

Theorem 4. *The real ε-pseudozero set of $p = \sum_{j\in J} a_j z^j \in \mathcal{P}^n(\mathbf{R})$ verifies*

$$Z_\varepsilon^R(p) = Z(p) \cup \left\{ z \in \mathbf{C}^n \backslash Z(p) : d(G_R(z), \mathbf{R}G_I(z)) \ge \frac{1}{\varepsilon} \right\}$$

where $G_R(z)$ and $G_I(z)$ are the real and imaginary parts of

$$G(z) = \frac{1}{p(z)}(\dots, z^j, \dots, j \in J)^T, \ z \in \mathbf{C}^n \backslash Z(p).$$

To compute the real ε-pseudozero set $Z_\varepsilon^R(p)$, we only have to evaluate the distance $d(G_R(z), \mathbf{R}G_I(z))$. This quantity can be calculated easily for the 2-norm. If $\|\cdot\|_2$ denotes the 2-norm and $\langle\cdot,\cdot\rangle$ the corresponding inner product, we have

$$d(x, \mathbf{R}y) = \begin{cases} \sqrt{\|x\|_2^2 - \frac{\langle x,y\rangle^2}{\|y\|_2^2}} & \text{if } y \ne 0, \\ \|x\|_2 & \text{if } y = 0. \end{cases}$$

For the ∞-norm, we have

$$d(x, \mathbf{R}y) = \begin{cases} \min_{\substack{i=0:n \\ y_i \ne 0}} \|x - (x_i/y_i)y\|_1 & \text{if } y \ne 0, \\ \|x\|_1 & \text{if } y = 0. \end{cases}$$

This theorem can be immediately extended to systems of polynomials.

Corollary 5. *The real ε-pseudozero set of $P = \{p_1, \dots, p_k\}$, $k \in \mathbf{N}$ verifies*

$$Z_\varepsilon^R(P) = \bigcup_{l=1}^{k} \left(Z(p_l) \cup \left\{ z \in \mathbf{C}^n \backslash Z(p_l) : d(G_R^l(z), \mathbf{R}G_I^l(z)) \ge \frac{1}{\varepsilon} \right\} \right)$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

$$G^l(z) = \frac{1}{p_l(z)}(\dots, z^j, \dots, j \in J_l)^T, \ z \in \mathbf{C}^n \backslash Z(p_l).$$

Visualization of pseudozero sets

The descriptions of $Z_\varepsilon(P)$ and $Z_\varepsilon^R(P)$ given by Theorem 1 and Theorem 4 enable us to compute, plot and visualize pseudozero set of multivariate polynomials. The pseudozero set is a subset of $\mathbf{C}^n$ which can only be seen by its projections on low dimensional spaces that is often $\mathbf{C}$.

For a given $v \in \mathbf{C}^n$, let $Z_\varepsilon(P, j, v)$ be the projection of $Z_\varepsilon(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon(P, j, v) = \left\{ z \in \mathbf{C}^n : z_i = v_i, \ i \ne j, \ \max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \le \varepsilon \right\},$$

where $\mathbf{z_l} := (\dots, |z|^j, \dots, j \in J_l)^T$. One way for visualizing $Z_\varepsilon(P, j, v)$ is to plot the values of the projection of

$$\operatorname{ps}(z) := \log_{10} \left(\max_{l=1,\dots,k} \frac{|p_l(z)|}{\|\mathbf{z_l}\|_*} \right)$$

over a set of grid points around v in z_j-space. In the same way, we define for a given $v \in \mathbf{C}^n$, $Z_\varepsilon^R(P, j, v)$ by the projection of $Z_\varepsilon^R(P)$ onto the z_j-space around v. Then, it follows that for $P = \{p_1, \dots, p_k\}$,

$$Z_\varepsilon^R(P, j, v) = \left\{ z \in \mathbf{C}^n : z_i = v_i, \ i \ne j, \ \max_{l=1,\dots,k} d(G_R^l(z), \mathbf{R}G_I^l(z))^{-1} \le \varepsilon \right\}$$

where $G_R^l(z)$ and $G_I^l(z)$ are the real and imaginary parts of

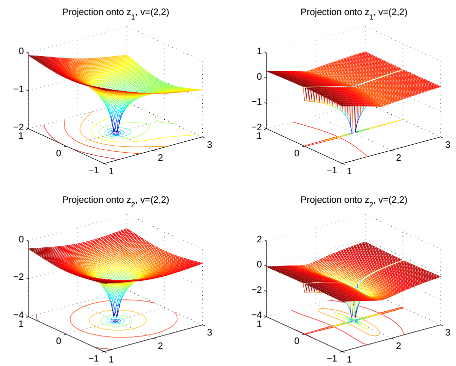
$$G^l(z) = \frac{1}{p_l(z)}(\dots, z^j, \dots, j \in J_l)^T, \ z \in \mathbf{C}^n \backslash Z(p_l).$$

One way for visualizing $Z_\varepsilon^R(P, j, v)$ is still to plot the values of the projection of

$$\operatorname{ps}^R(z) := \log_{10} \left(\max_{l=1,\dots,k} d(G_R^l(z), \mathbf{R}G_I^l(z))^{-1} \right)$$

over a set of grid points around v in z_j-space. We examine the following system using the 2-norm: two unit balls intersection at $(2, 2)$,

$$P_1 = \begin{cases} p_1 = (z_1 - 1)^2 + (z_2 - 2)^2 - 1, \\ p_2 = (z_1 - 3)^2 + (z_2 - 2)^2 - 1. \end{cases}$$



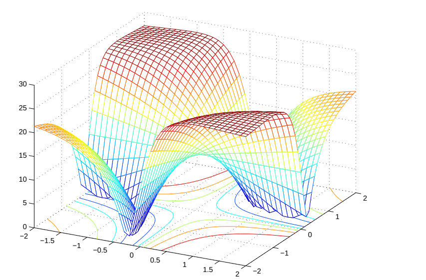
We can be only interested in the real zeros of a polynomial systems. In this case, we can only draw $\mathbf{R}^n \cap Z_\varepsilon^R(P)$. This is what is done with the following example still with the 2-norm,

$$P_2 = \begin{cases} p_1 = z_1^2 + z_2^2 - 1, \\ p_2 = 25 z_1 z_2 - 12. \end{cases}$$

In this Figure, we have computed the function

$$g(x, y) = \max_{l=1,2} \frac{p_l(x, y)}{\|\mathbf{z_l}\|_*},$$

where $\mathbf{z_l} := (\dots, |x + iy|^j, \dots, j \in J_l)^T$.



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