Symmetries and Dynamics of Discrete Systems

Talk at CASC 2007, Bonn, Germany

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Motivations

• Symmetry analysis of finite discrete systems can be complete

in contrast to continuous systems where only negligible small part of all thinkable symmetries is considered: point and contact Lie, Bäcklund and Lie–Bäcklund, sporadic instances of non-local symmetries etc.

• Many hints from quantum mechanics and quantum gravity that discreteness is more suitable for physics at small distances

than continuity which arises only as a logical limit in considering large collections of discrete structures

Discrete Relations and Discrete Dynamical Systems

Any collection of discrete points taking values in finite sets possesses some kind of locality:

such collection has a structure of abstract simplicial complex
Special cases are

- systems of polynomial equations over finite fields
- cellular automata

In this talk we consider relations on graphs — 1D simplicial complexes

Two types of discrete dynamical systems — discrete relations evolving in discrete time:

- state of deterministic system at any point of time is function of previous states (we consider cellular automata with symmetric local rules)
- in non-deterministic system transition from one state to any other state is possible with some probability (we consider mesoscopic lattice models — some ensembles of discrete relations)

Lattices and Their Symmetries

- Space of discrete dynamical system is lattice $\Gamma \equiv k$-valent graph
- Symmetry of lattice Γ is graph automorphism group $\mathrm{Aut}\left(\Gamma\right)$
- We assume $\mathrm{Aut}\left(\Gamma\right)$ acts transitively on vertices $V(\Gamma)$ of lattice: very idea of “space” implies possibility to reach any point by “moves” in the space

- In applications it is assumed often that lattice is embedded in some continuous space — in this case notion of ‘dimension’ of lattice makes sense
- In our context such embeddings are of little importance — we shall use them for visualization only

Functions on Lattices and Their Orbits

- $Q = \{0, \ldots, q-1\}$ is set of values of lattice vertices
- $\Sigma = Q^{\Gamma}$ is space of Q-valued functions on Γ
- $\mathrm{Aut}(\Gamma)$ acts non-transitively on Σ splitting this space into orbits of different sizes $\Sigma = \bigcup_{i=1}^{N_{orbits}} O_i$

  Action: $(g\varphi)(x) = \varphi(g^{-1}x)$

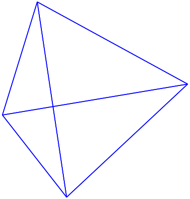
  $x \in V(\Gamma),\ \varphi(x) \in \Sigma,\ g \in \mathrm{Aut}(\Gamma)$
- Burnside's lemma counts total number of orbits

$$N_{orbits} = \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{g \in \mathrm{Aut}(\Gamma)} q^{N_{cycles}^{g}},$$

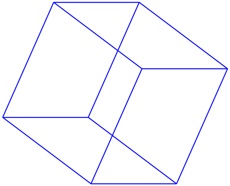
N_{cycles}^{g} is number of cycles in group element g

Examples of Lattices

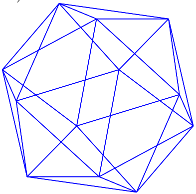
Tetrahedron



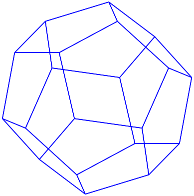
Hexahedron (= Graphene 4×2)



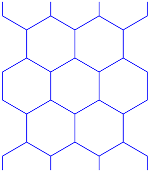
Icosahedron



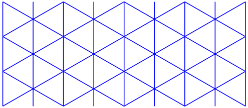
Dodecahedron



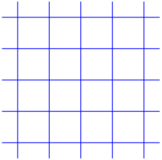
Graphene 6×4



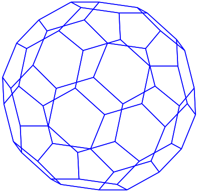
Triangular 4×6



Square 5×5



Buckyball

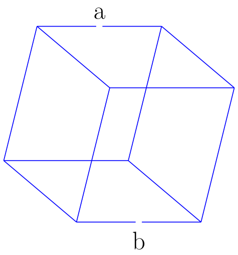


Cube Is “Smallest Graphene” (Graphene 4×2)

Embeddings of graph of Hexahedron form

4-gonal (6 tetragons) regular tiling of sphere $\mathbb{S}^2$ and

6-gonal (4 hexagons) regular tiling of torus $\mathbb{T}^2$



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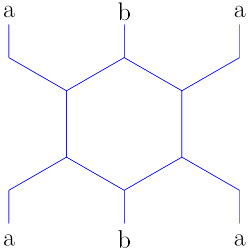

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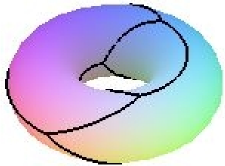


Some Quantitative Characteristics

of lattices, their groups and orbits

Lattice	$\lvert V(\Gamma)\rvert$	$\lvert \mathrm{Aut}(\Gamma)\rvert$	$\Omega = q^{\lvert V(\Gamma)\rvert}$	N_{orbits}
Tetrahedron	4	24	16	5
Hexahedron	8	48	256	22
Icosahedron	12	120	4096	82
Dodecahedron	20	120	1048576	9436
Graphene 6×4 Torus	24	48	16777216	355353
Graphene 6×4 Klein bottle	24	16	16777216	1054756
Triangular 4×6	24	96	16777216	180070
Square 5×5	25	200	33554432	172112
Buckyball	60	120	1152921504606846976 $\approx 10^{18}$	9607679885269312 $\approx 10^{16}$

Klein bottle arrangement is non-transitive (2 orbits of sizes 8 and 16)

We discard such graphs as counterintuitive nonhomogeneous spaces

C Program

Input

- Graph $\Gamma = \{N_1, \ldots, N_n\}$
  N_i is neighborhood of ith vertex
- Cellular automata branch:
  - Set of local rules $R = \{r_1, \ldots, r_m\}$
    r_i is bit representation of ith rule
- Lattice model branch:
  - Hamiltonian of the model
- Control parameters

C Program

Program produces

- automorphism group $\mathrm{Aut}\,(\Gamma)$
- Cellular automata branch:
  - phase portraits of automata modulo $\mathrm{Aut}(\Gamma)$ for all rules from R
    Manipulating control parameters we can
    - select automata with specified properties, e.g., reversibility, Hamiltonian conservation etc.
    - search automata producing specified structures, e.g., limit cycles, isolated cycles, "Gardens of Eden", "Spaceships" etc.
- Lattice model branch:
  - partition function and other characteristics of the model
  - search of phase transitions

General Dynamical Principle Induced By Symmetry

For Any Deterministic Dynamical System

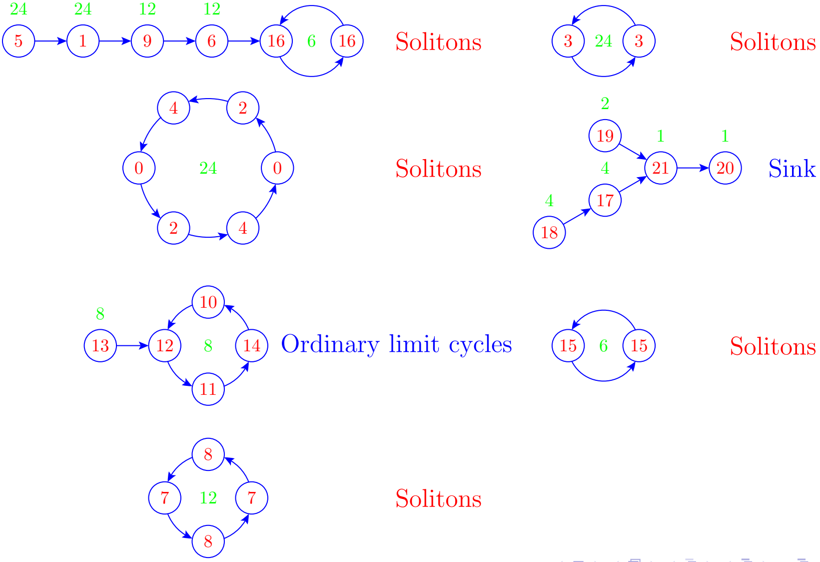
- trajectories can be directed only from larger orbits to smaller orbits or to orbits of the same size

- periodic trajectories must lie within the orbits of equal size

- Meaning: any isolated system may only lose information in its evolution — analog of “Second Law of Thermodynamics”
- Simple consequence of the fact that
  “deterministic” means “functional”

Group Origin of Emergent Structures: typical phase portrait

modulo $\mathrm{Aut}(\Gamma)$ contains soliton-like structures — “spaceships”



Eventually (Part of) Dynamics Is Reduced To Group Actions

State $\varphi(x) \in O_i$ in cycle starting at moment 0 evolves after some time t into state $\varphi_t(x) = A_t(\varphi(x))$ in the same orbit O_i

Hence:

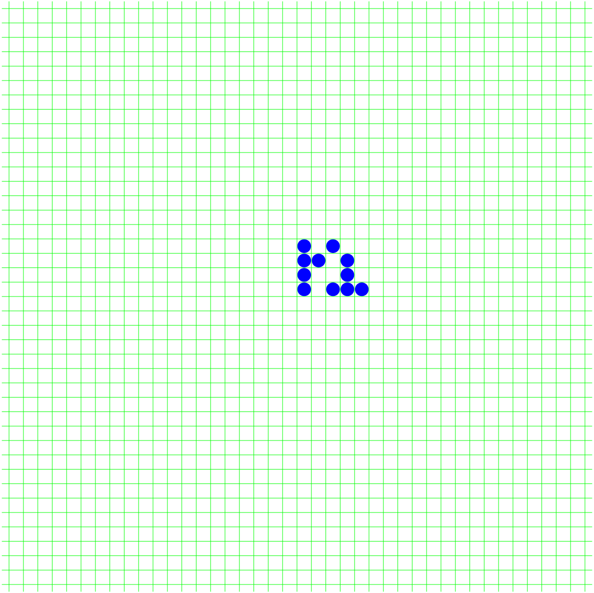
Evolution operator A_t can be replaced by group action

$$\varphi_t(x) = A_t(\varphi(x)) = \varphi\left(g_t^{-1}x\right)$$

i.e., initial shape $\varphi(x)$ is reproduced via some movement in space — formation of soliton-like structure

Galilei group analogy: many equations of mathematical physics have running wave solutions $\varphi(x - vt)$ $\left(= \varphi\left(g_t^{-1}x\right)\right.$ for Galilei group$\left.\right)$

Conway's Life. Typical Evolution Producing Gliders



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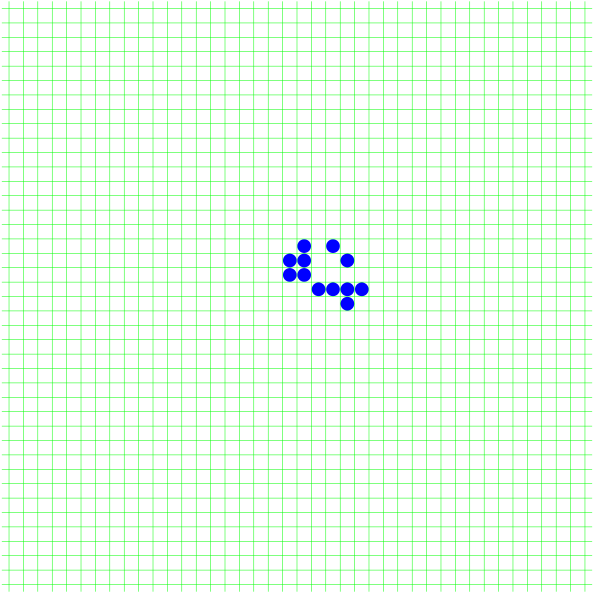

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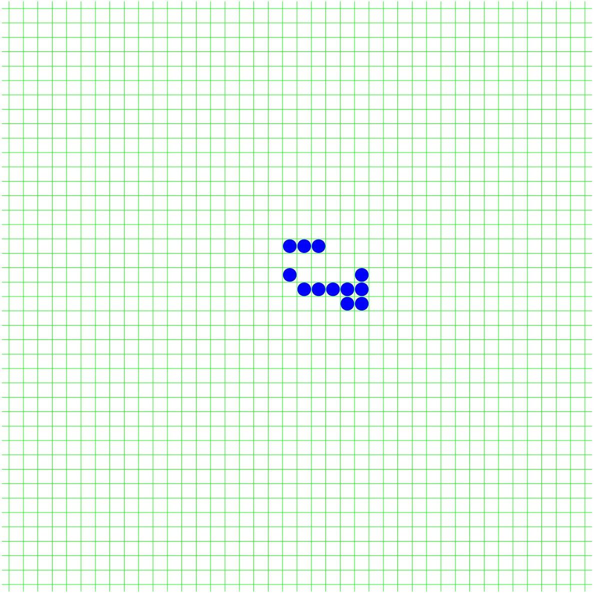

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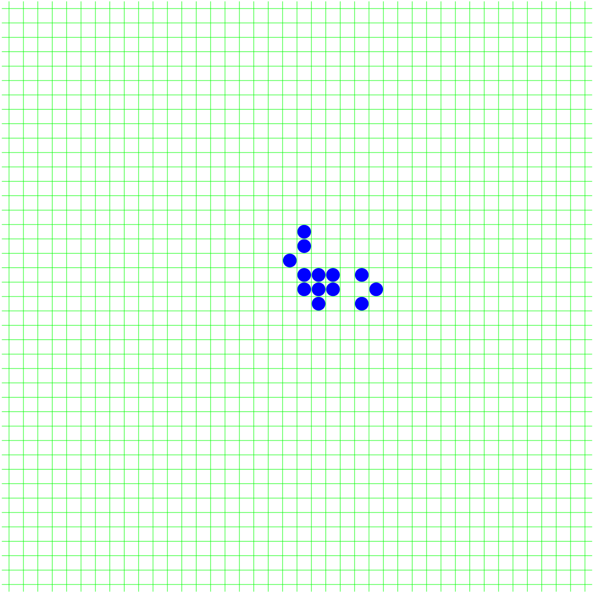

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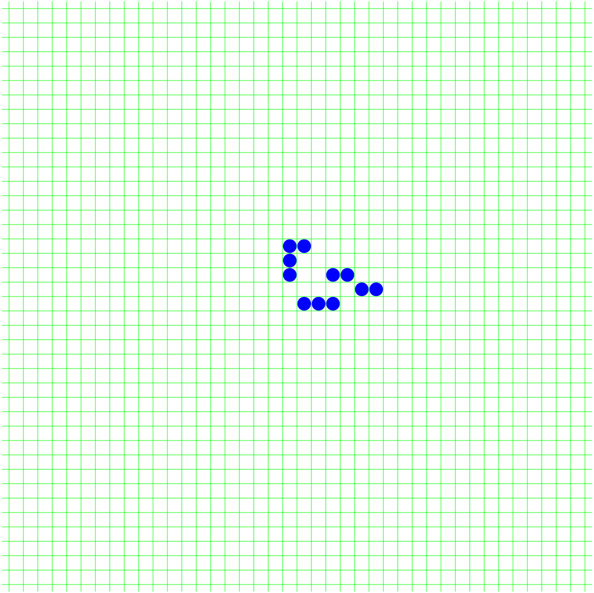

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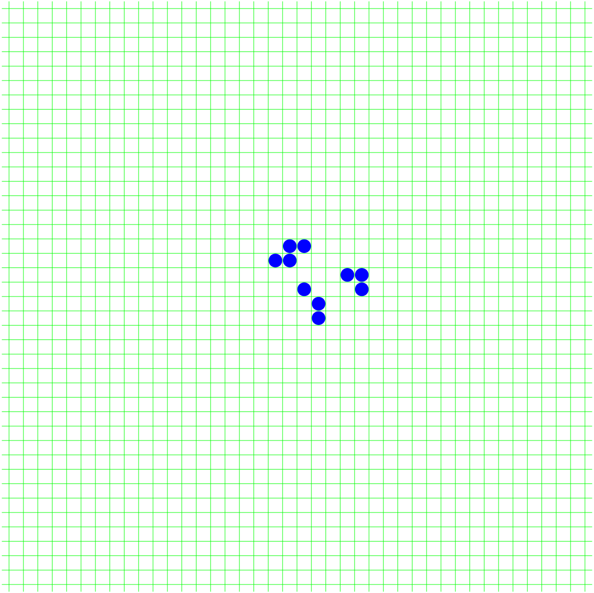

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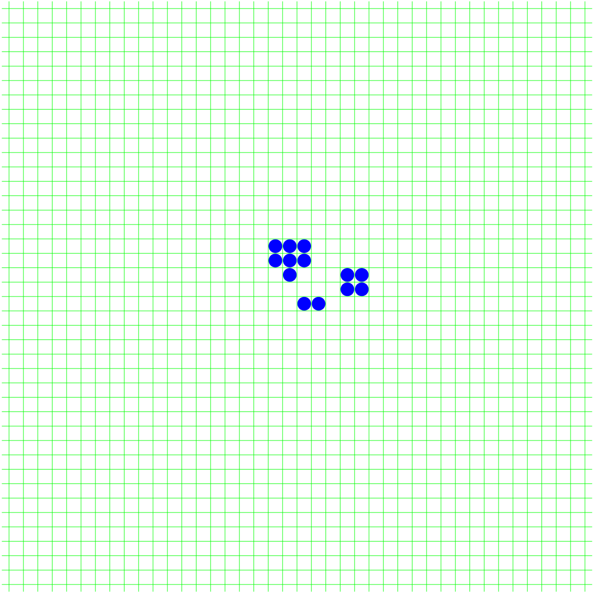

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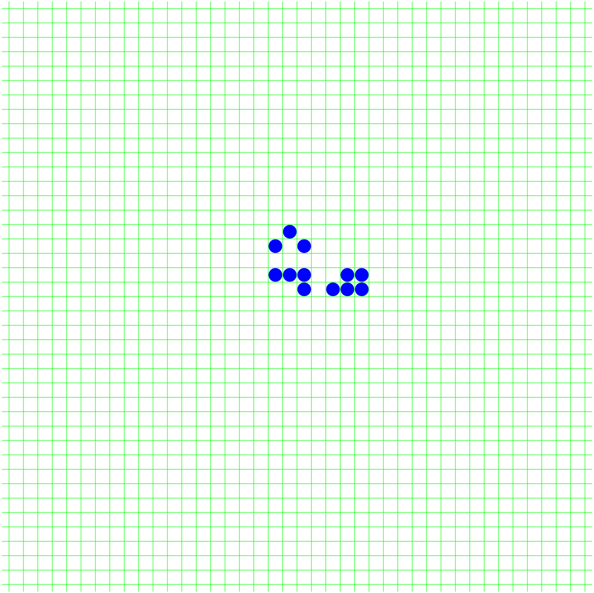

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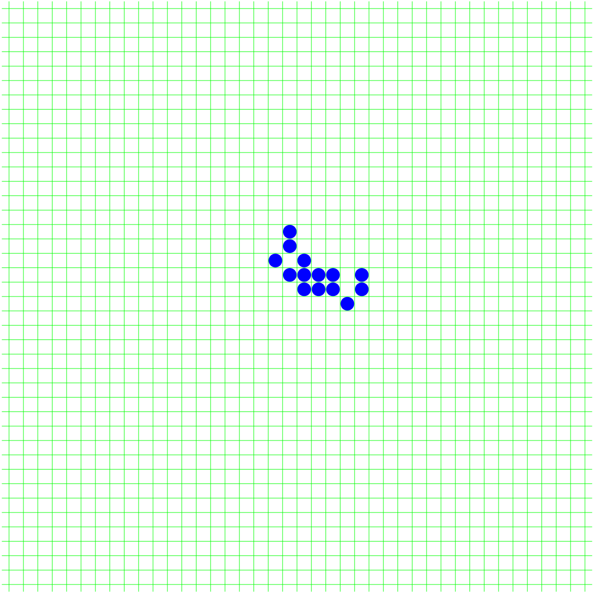

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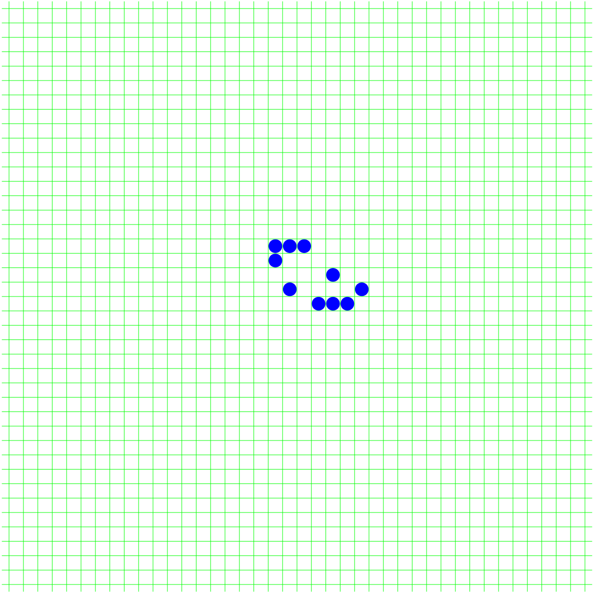

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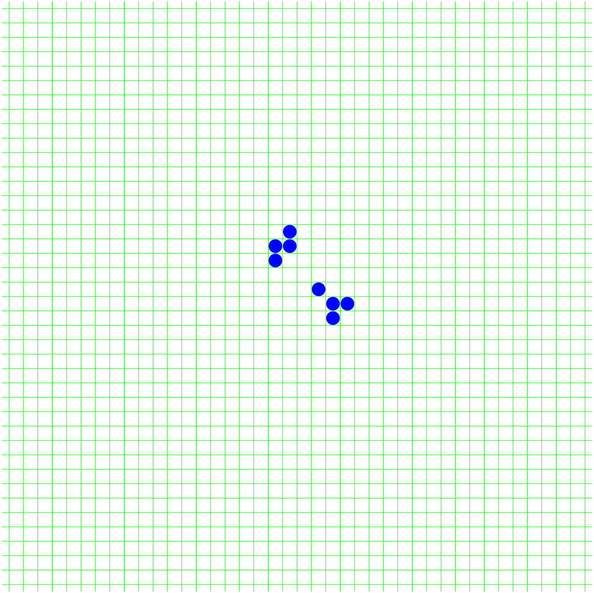

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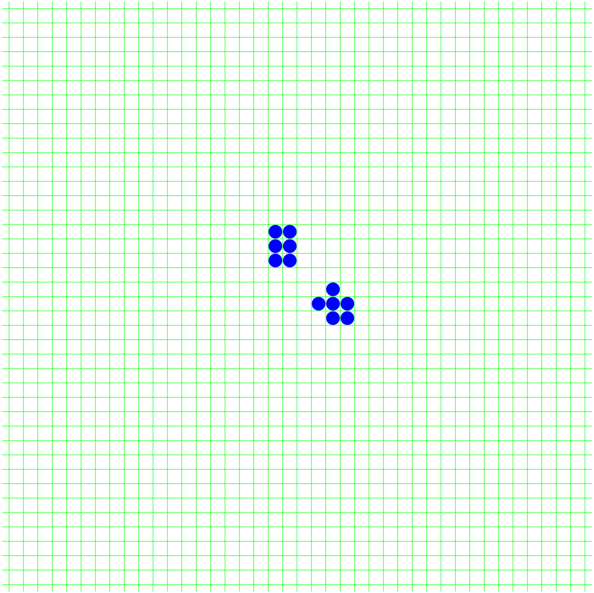

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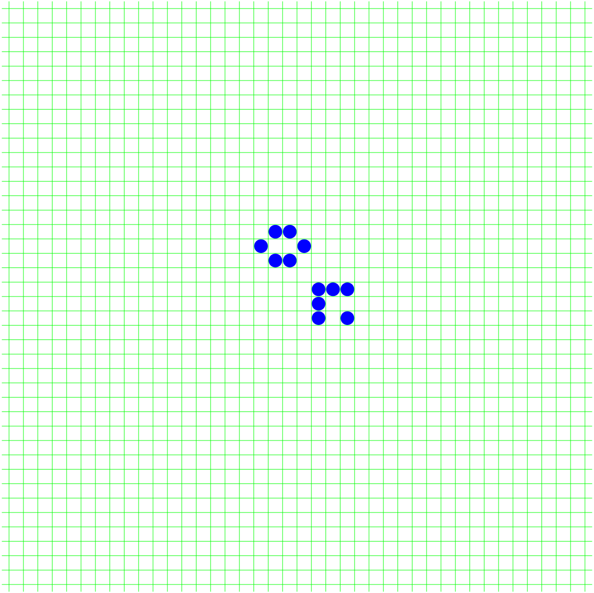

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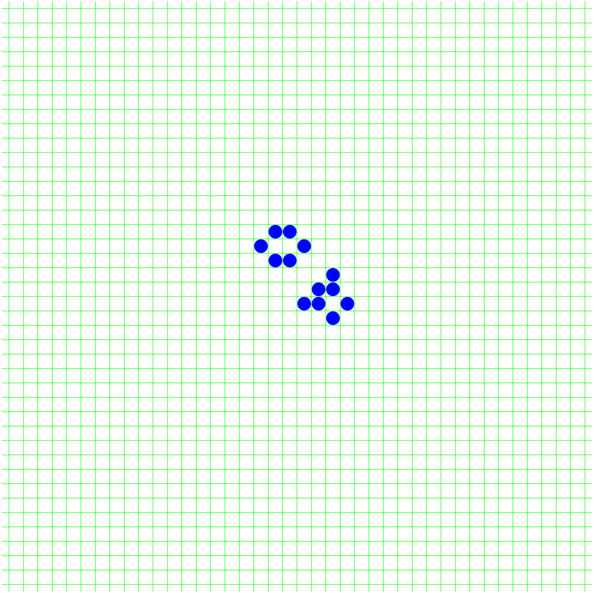

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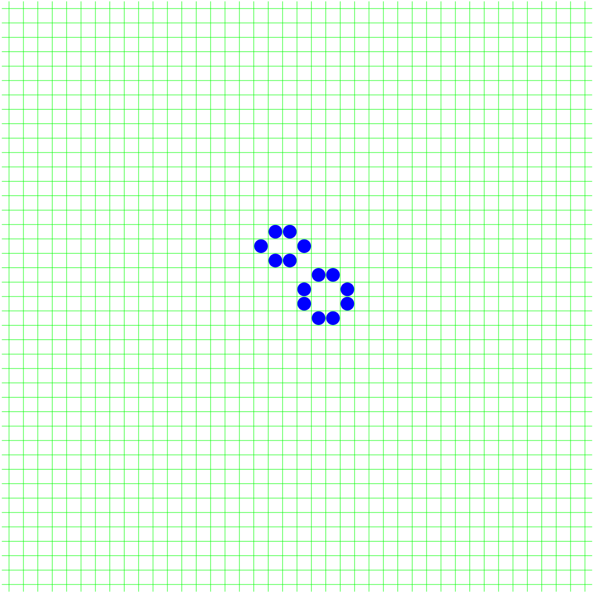

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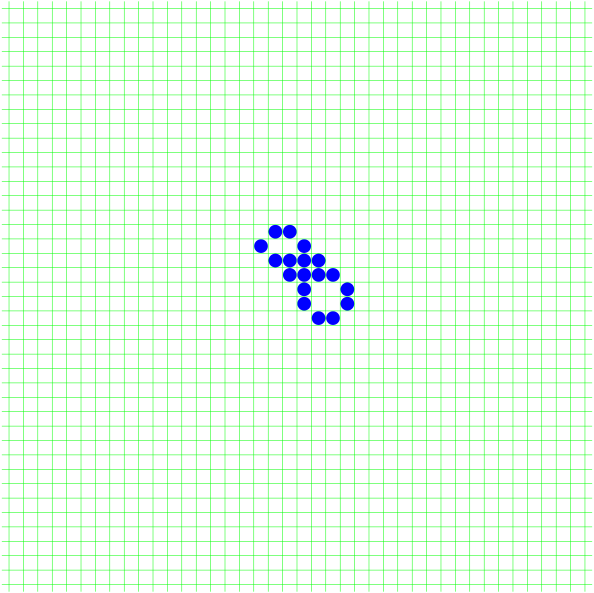

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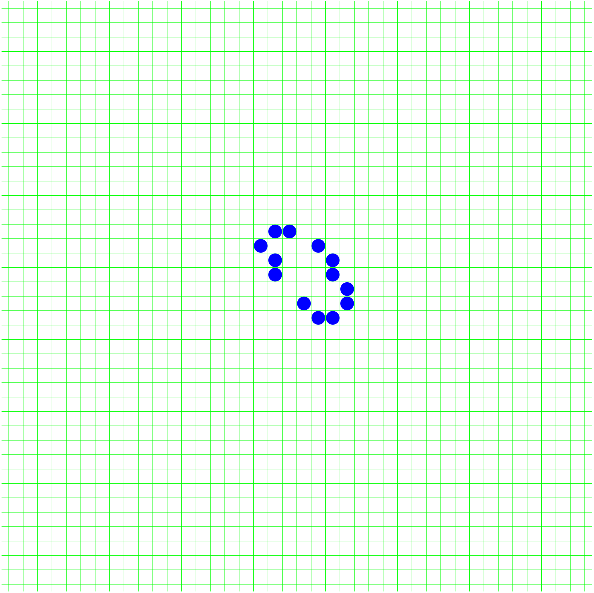

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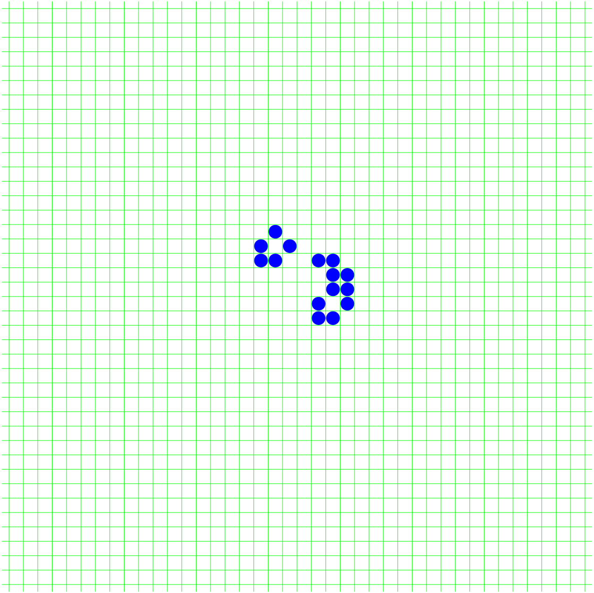

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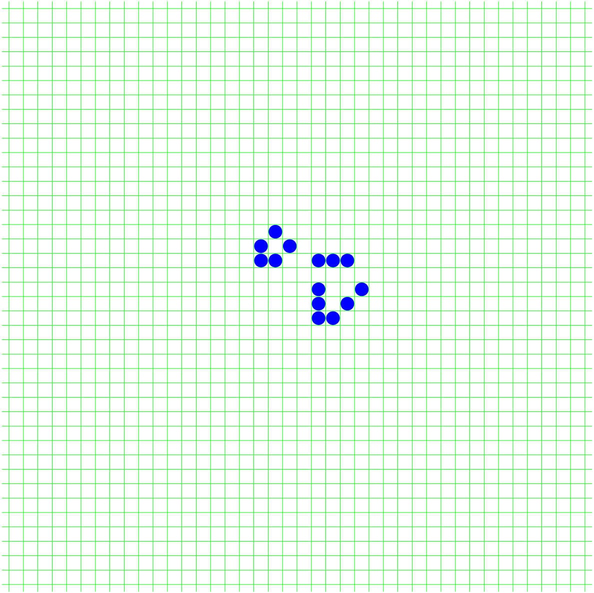

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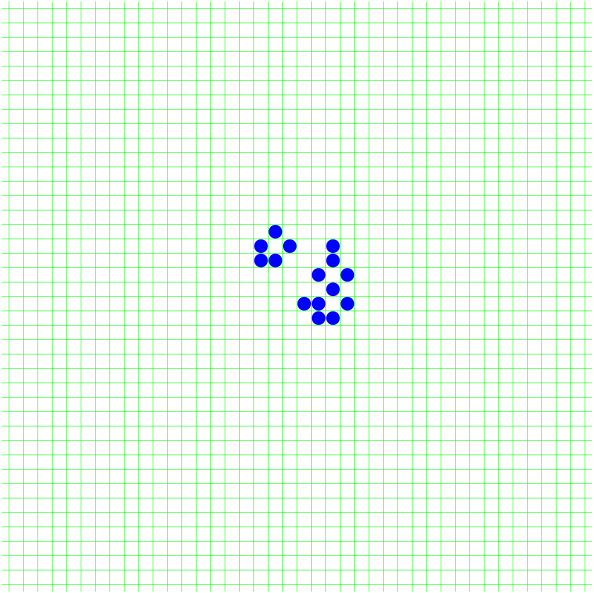

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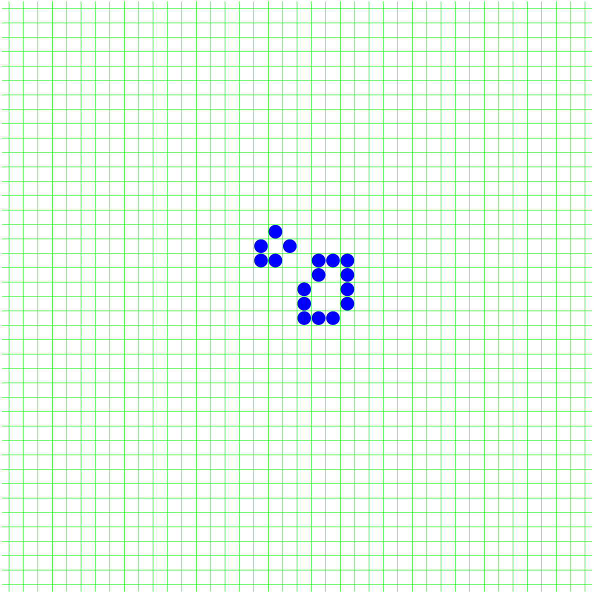

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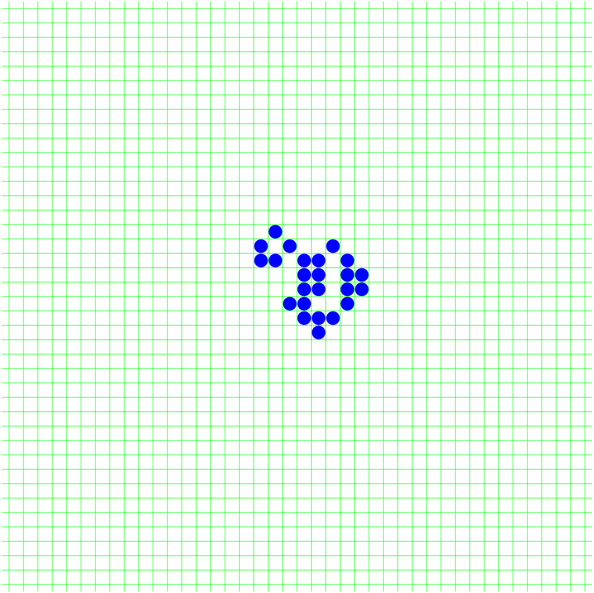

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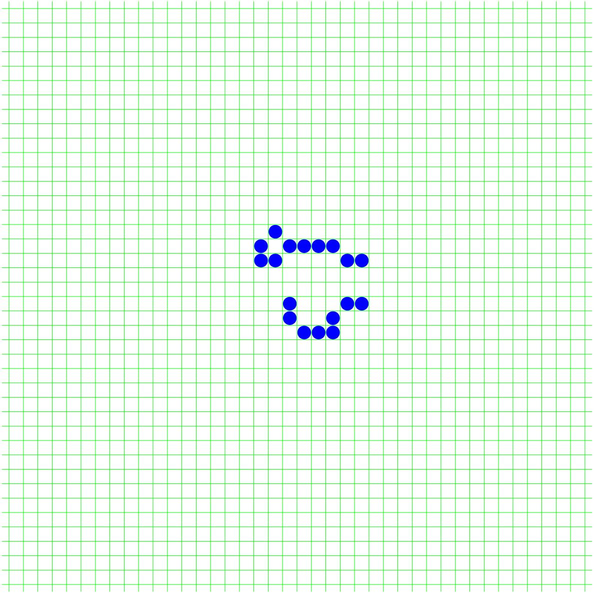

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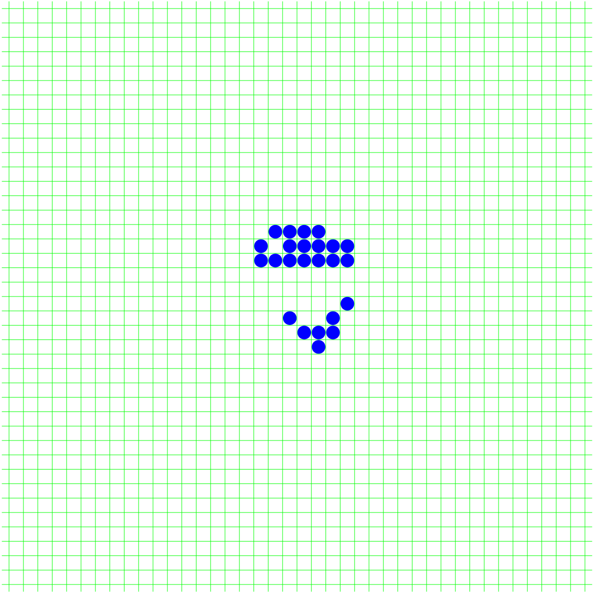

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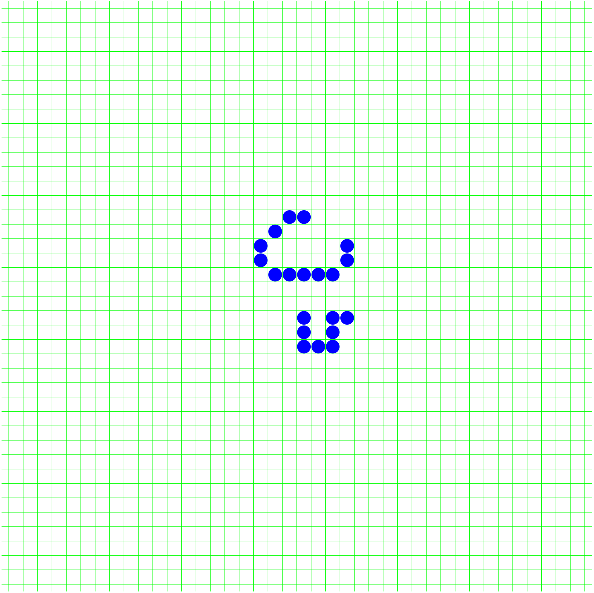

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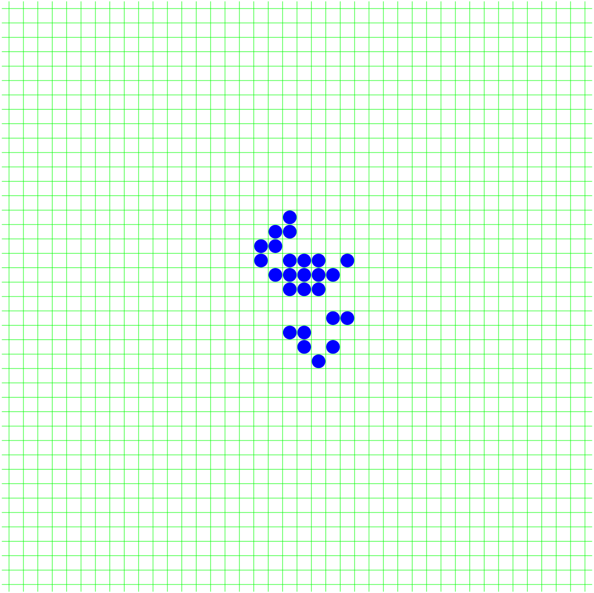

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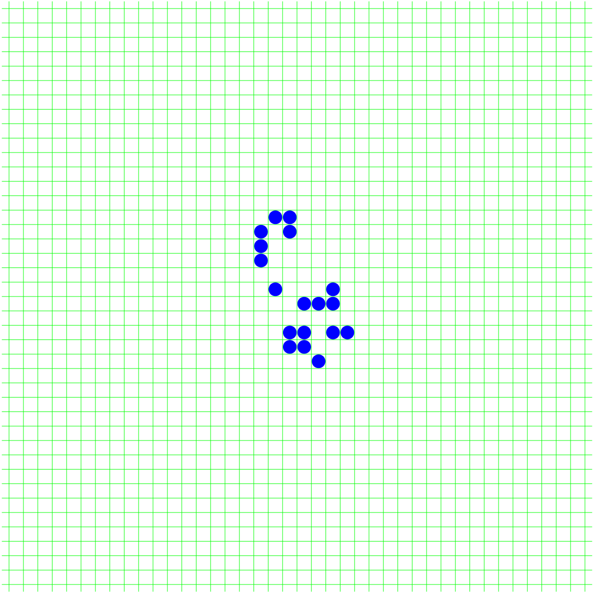

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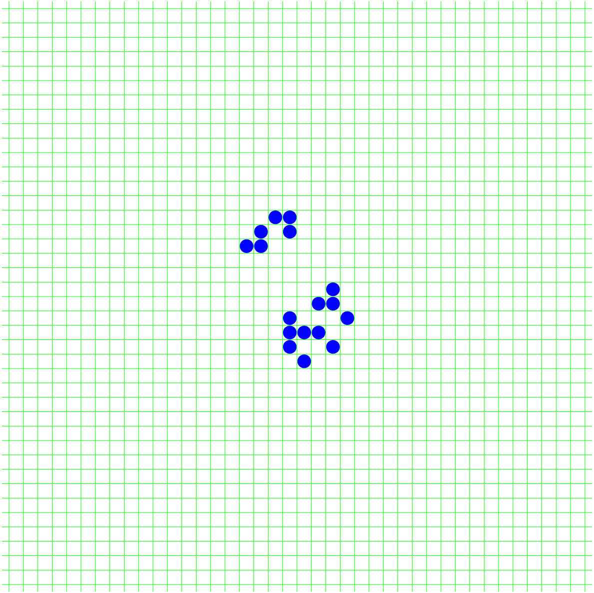

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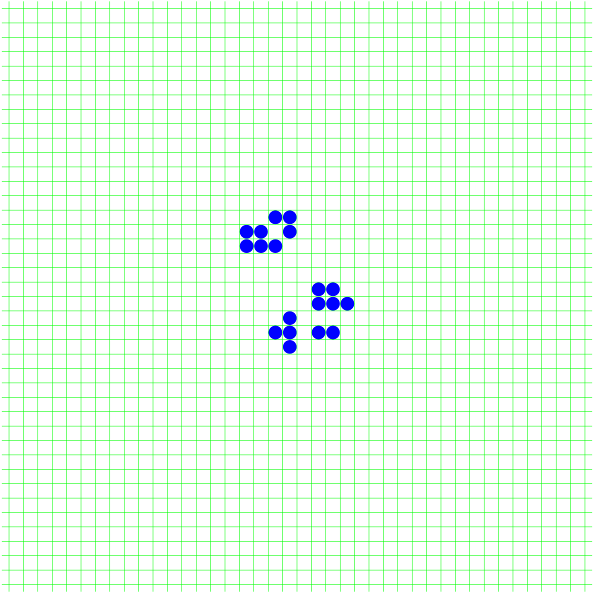

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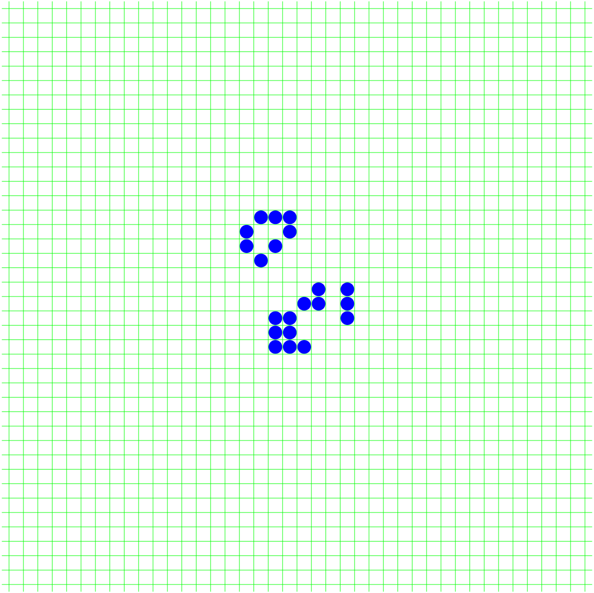

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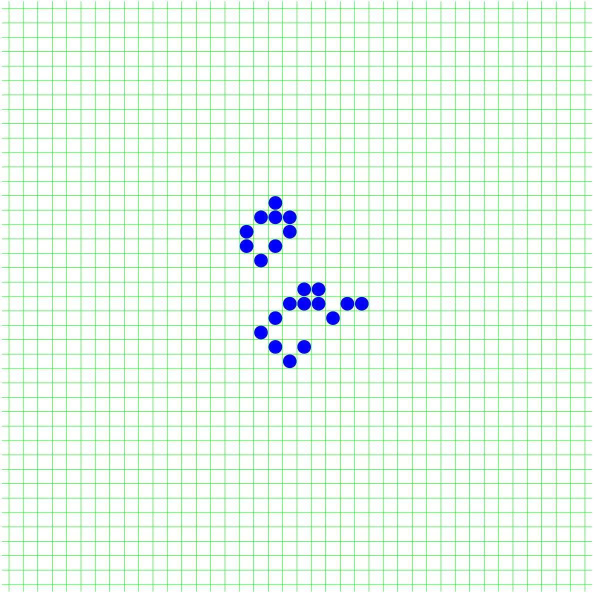

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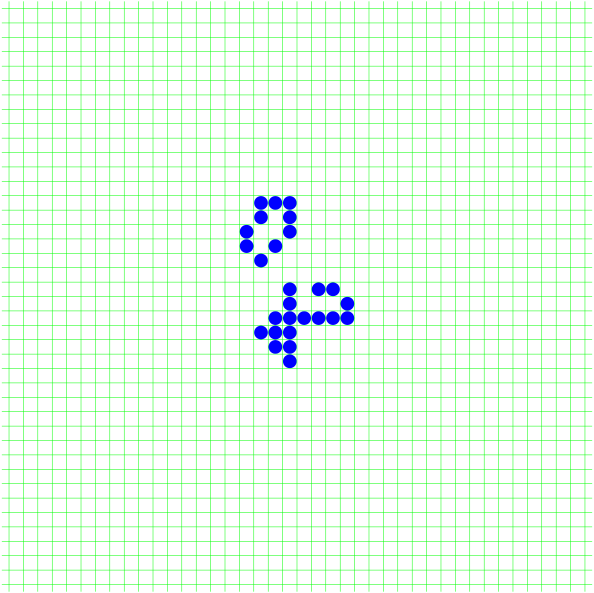

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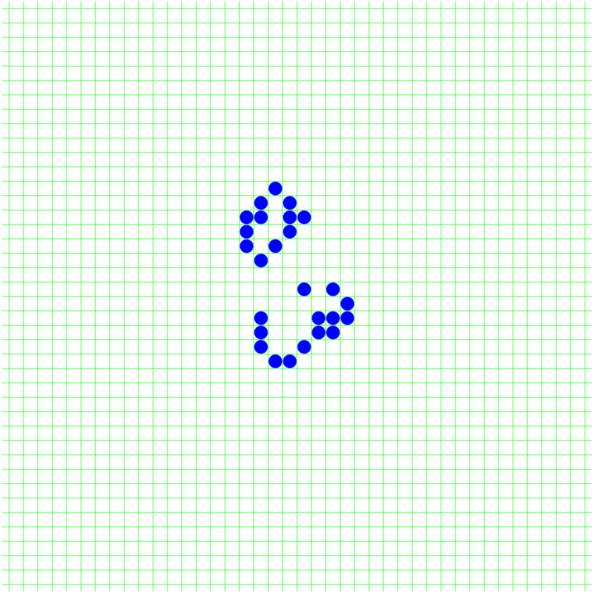

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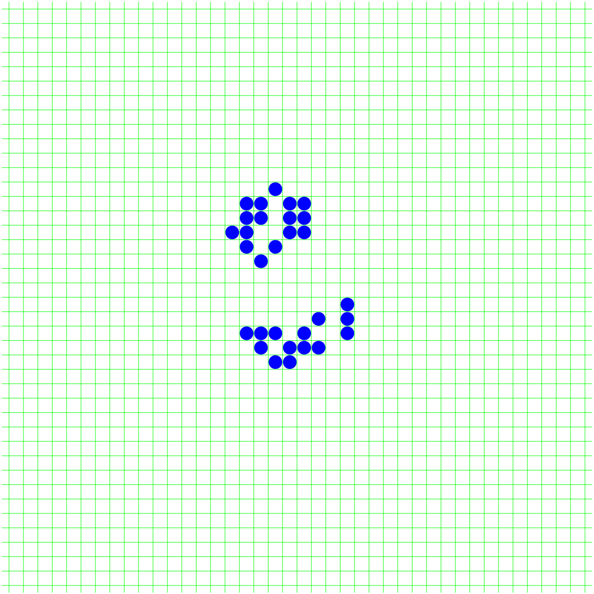

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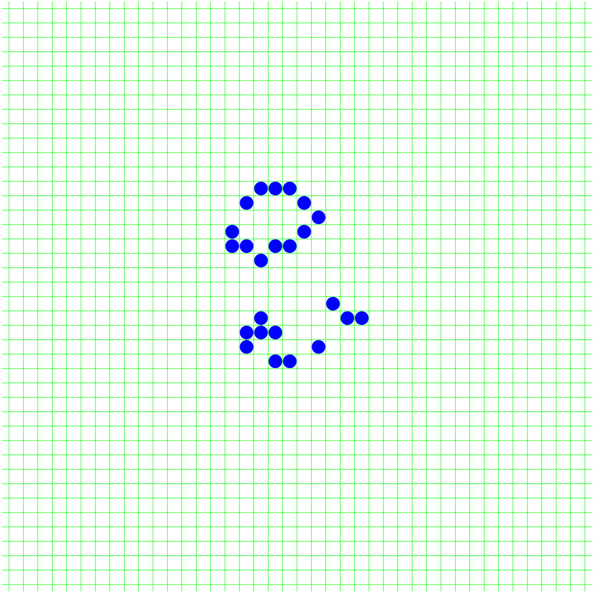

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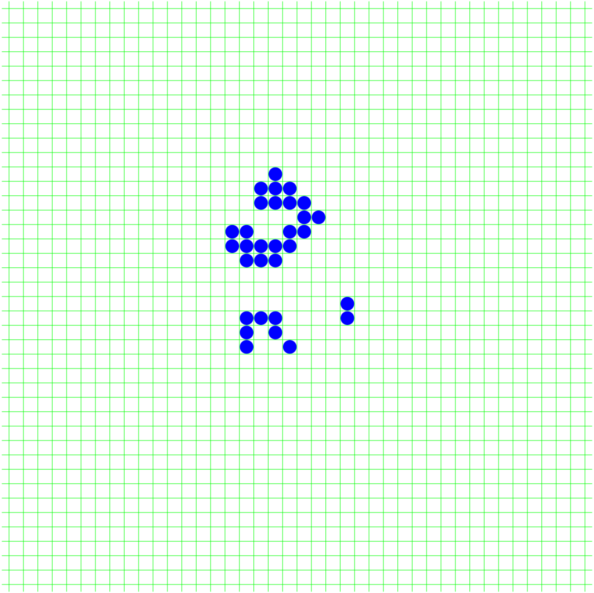

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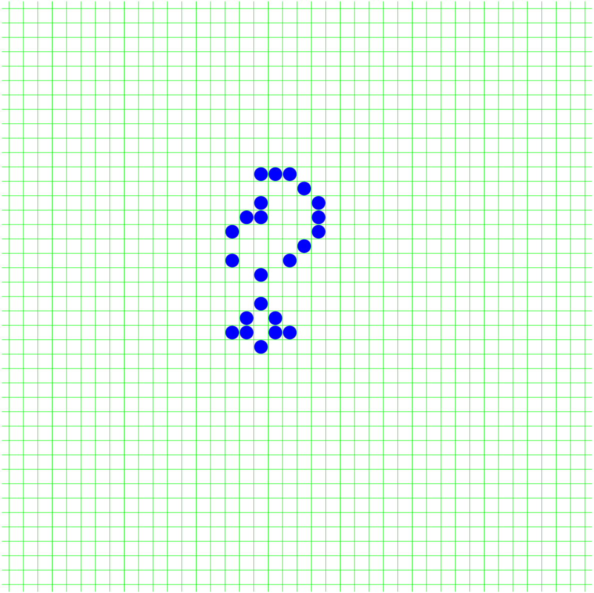

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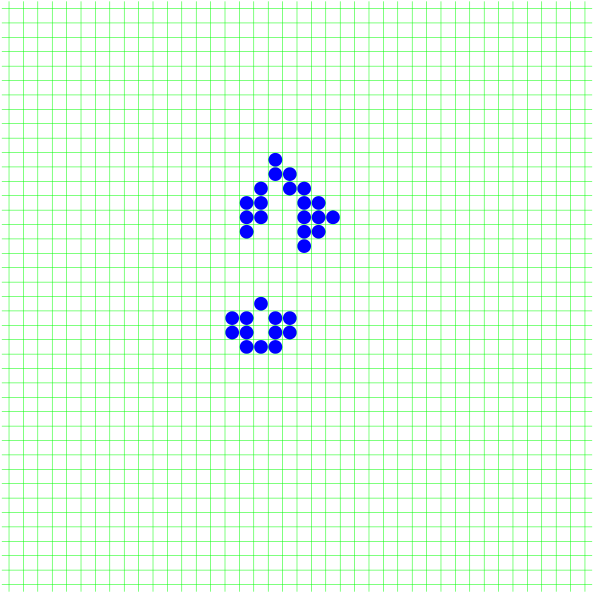

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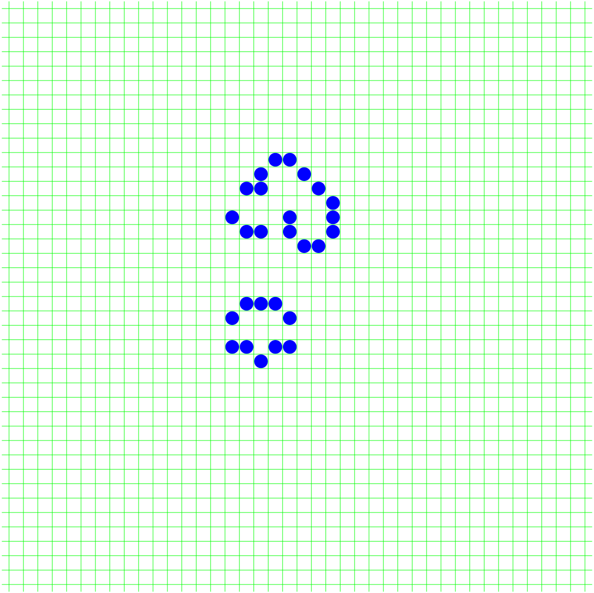

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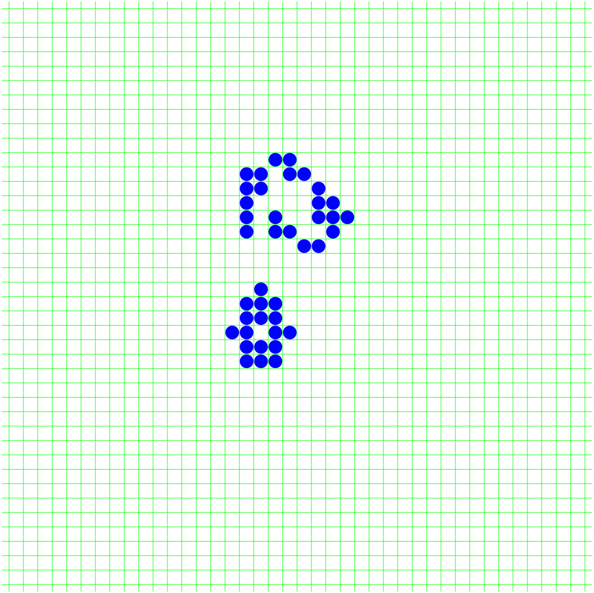

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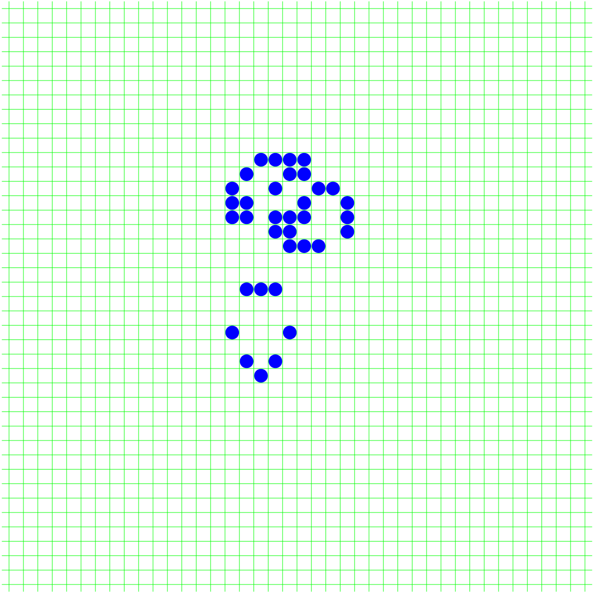

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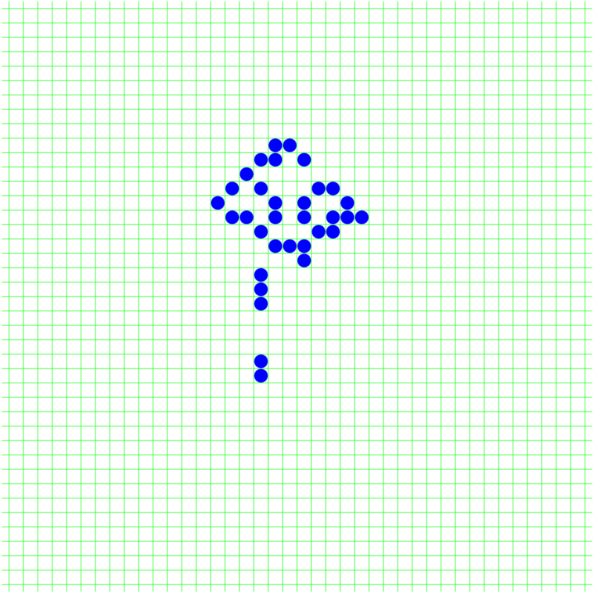

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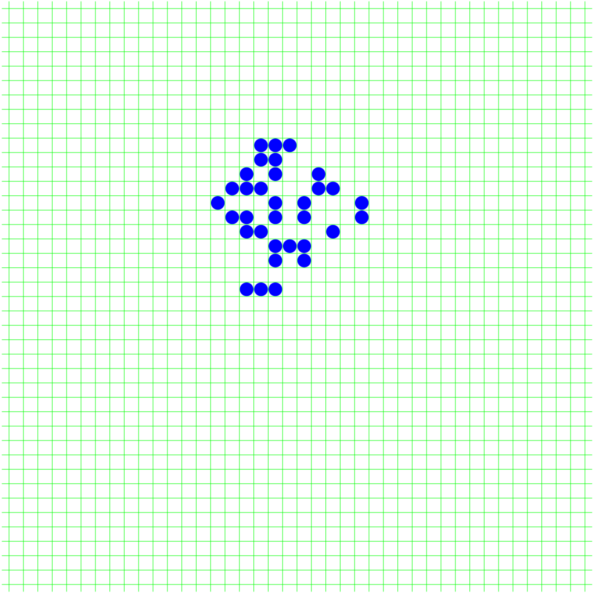

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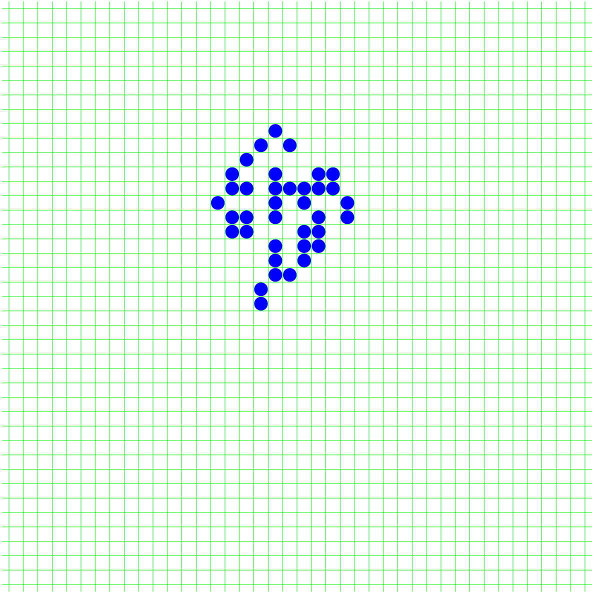

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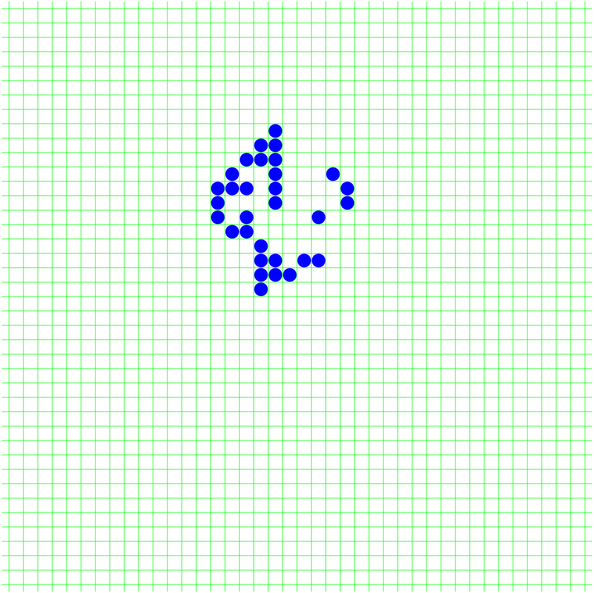

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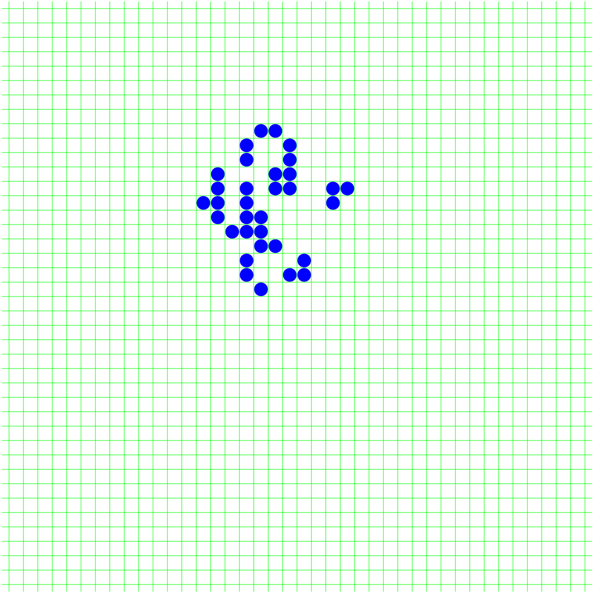

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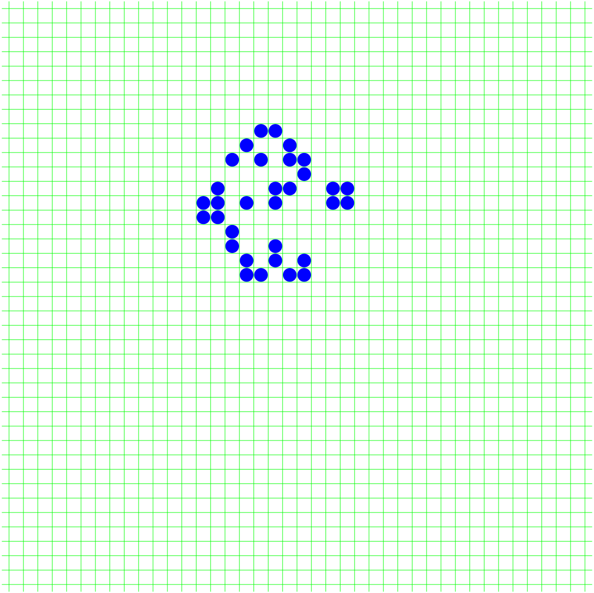

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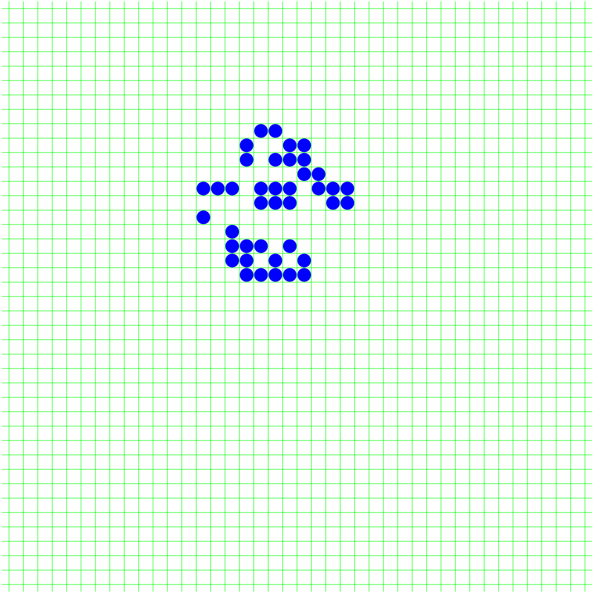

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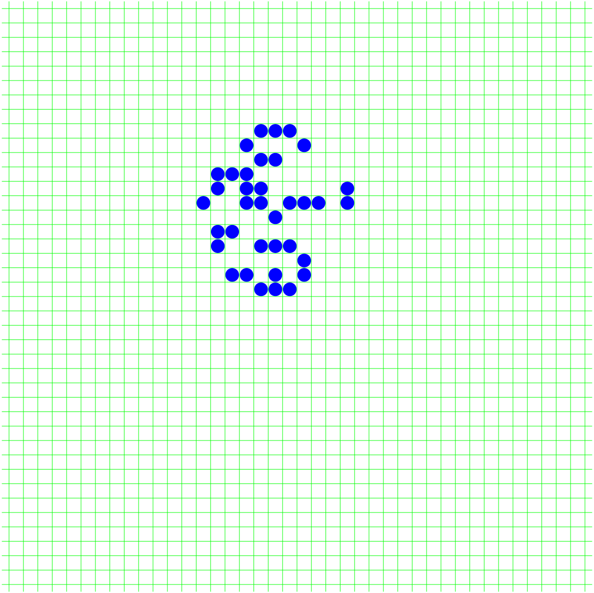

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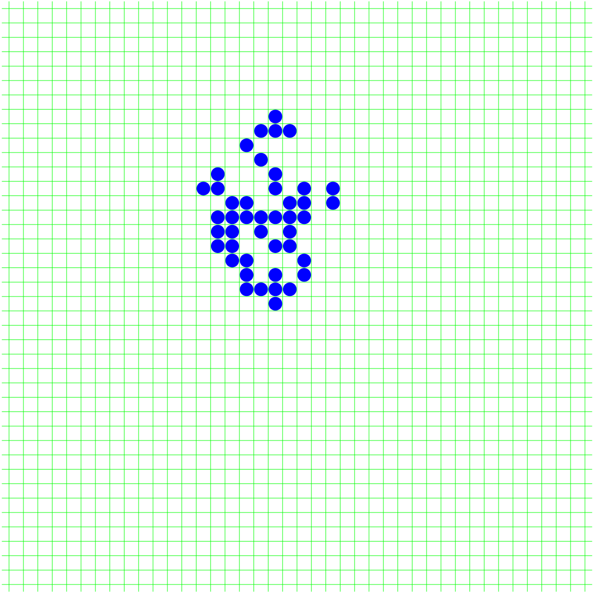

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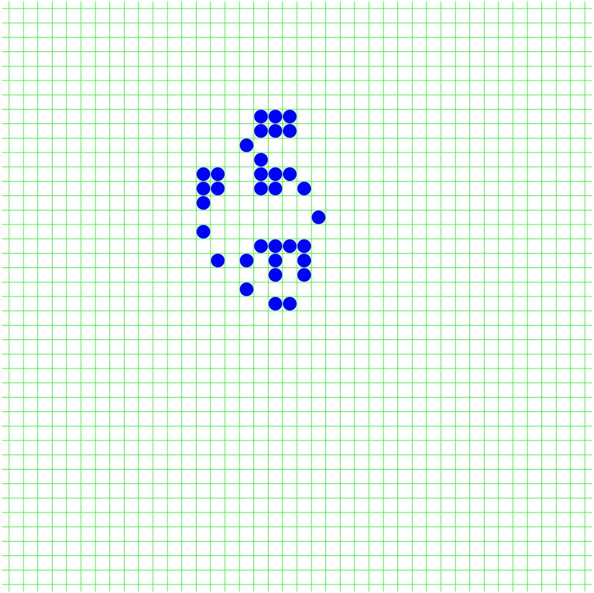

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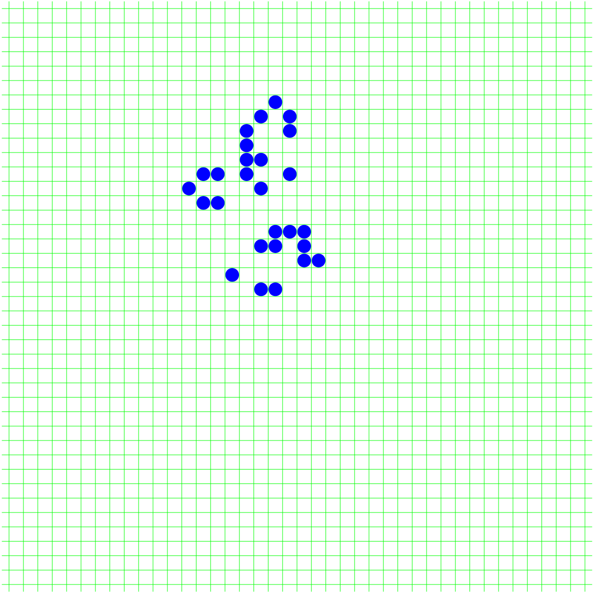

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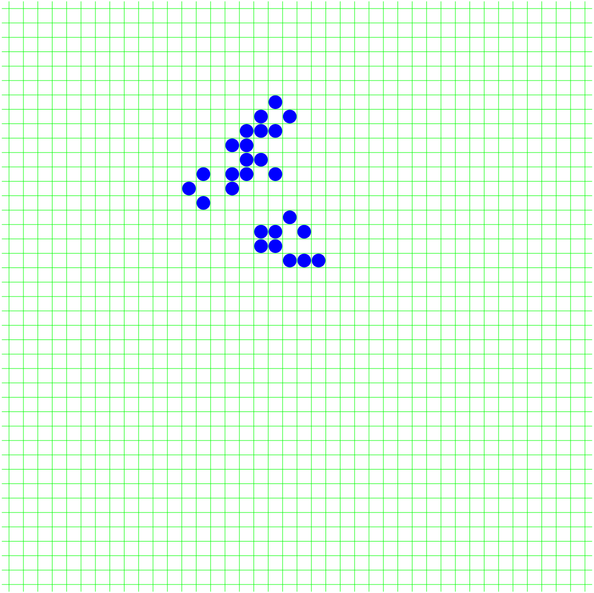

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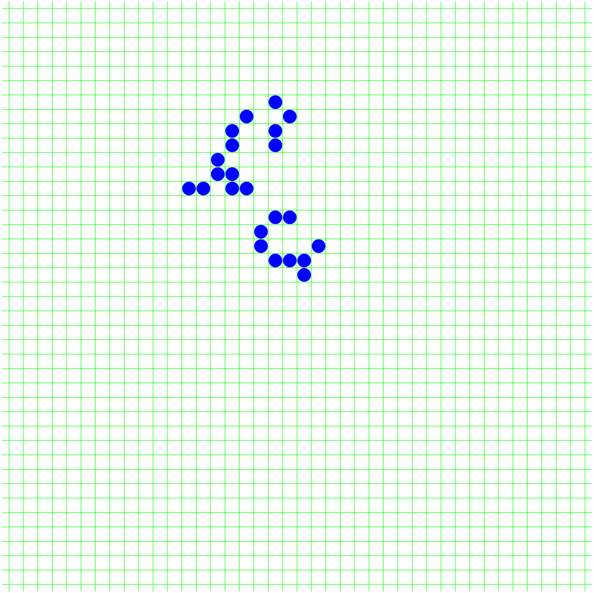

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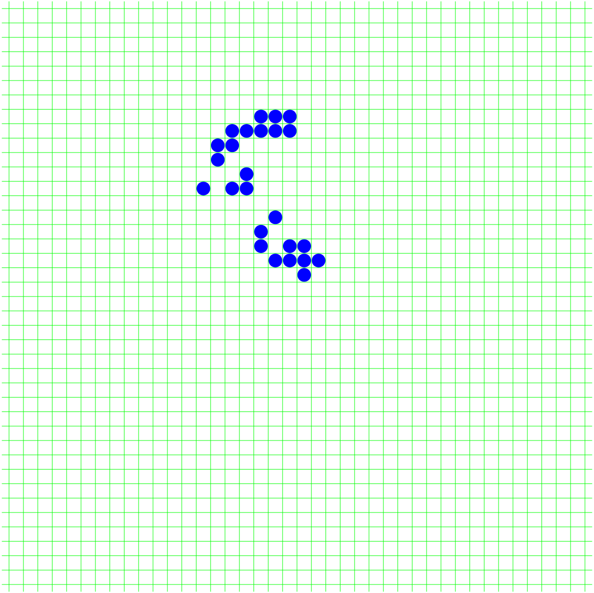

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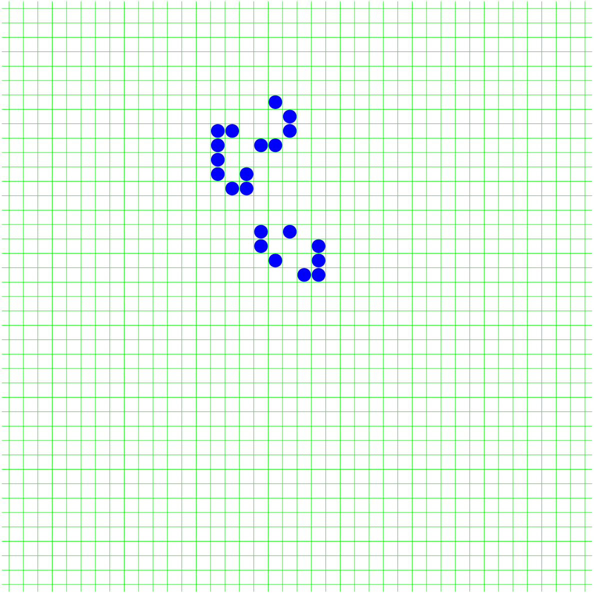

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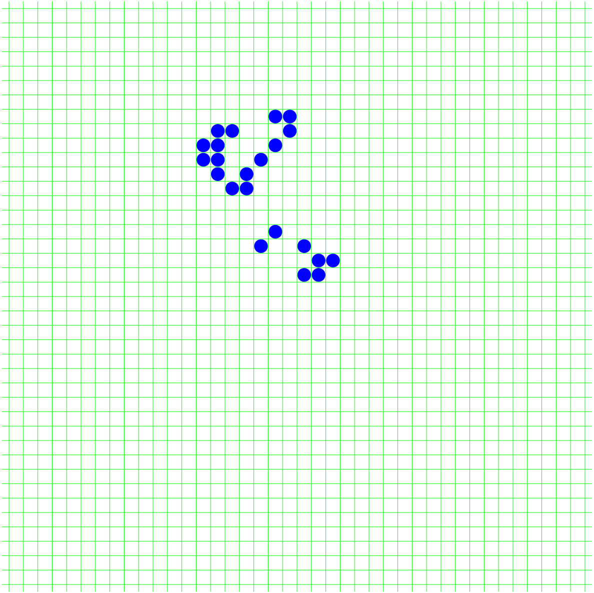

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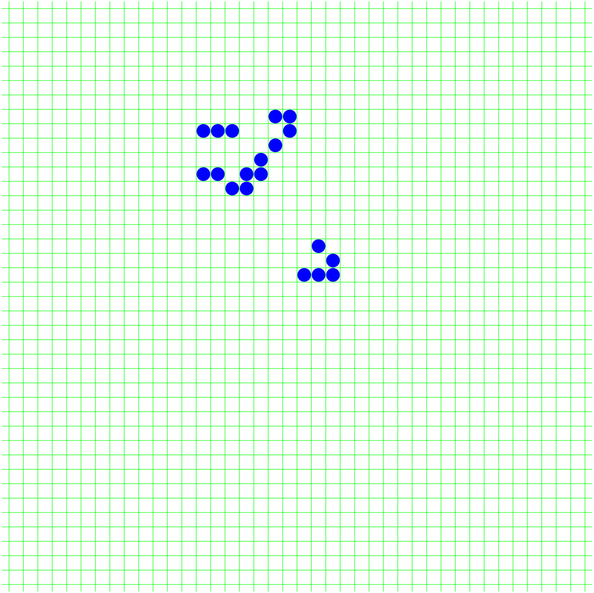

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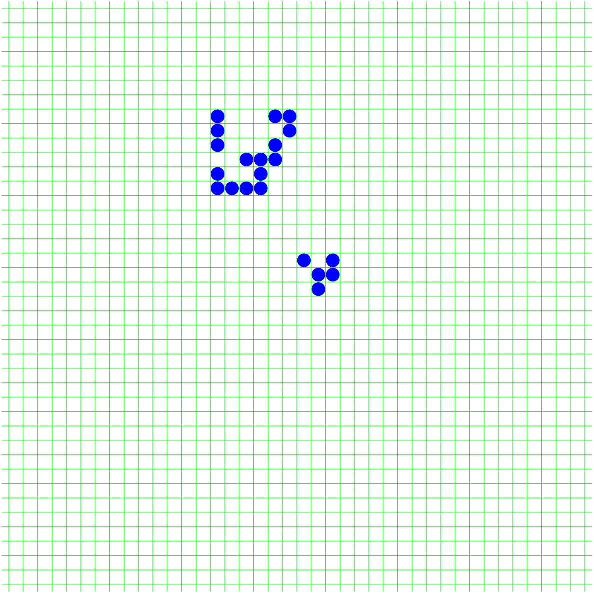

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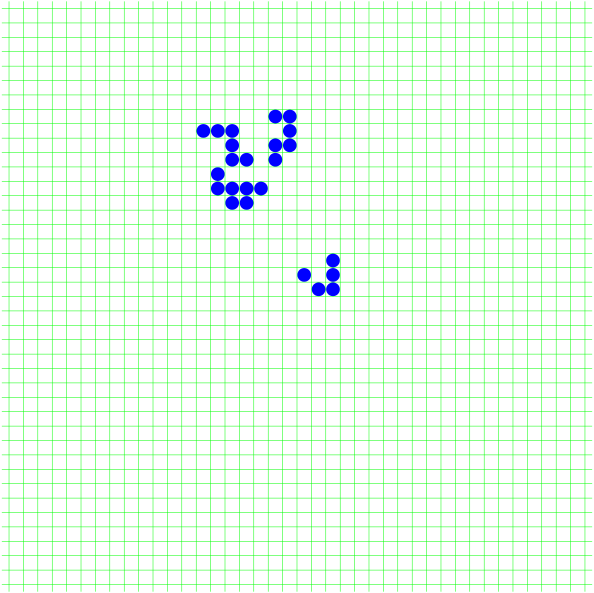

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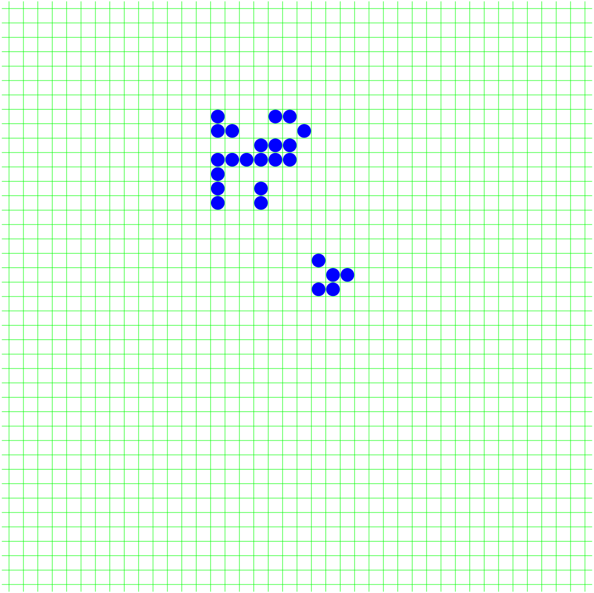

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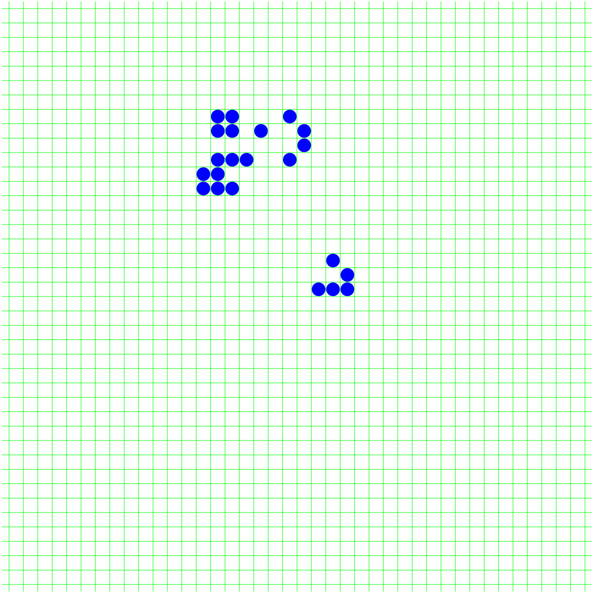

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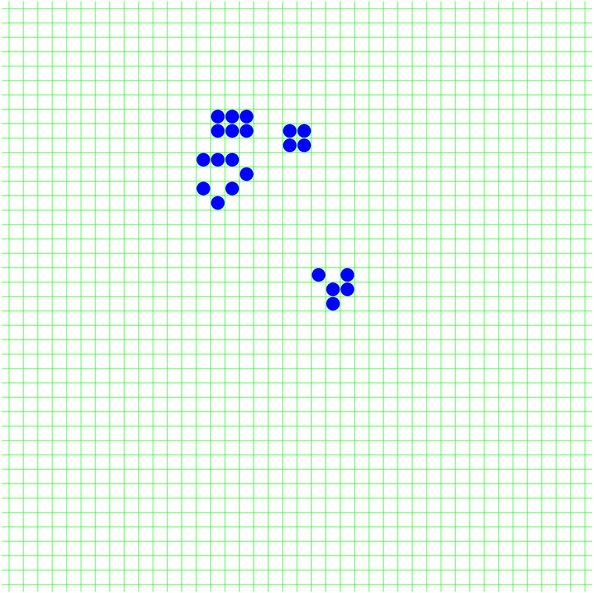

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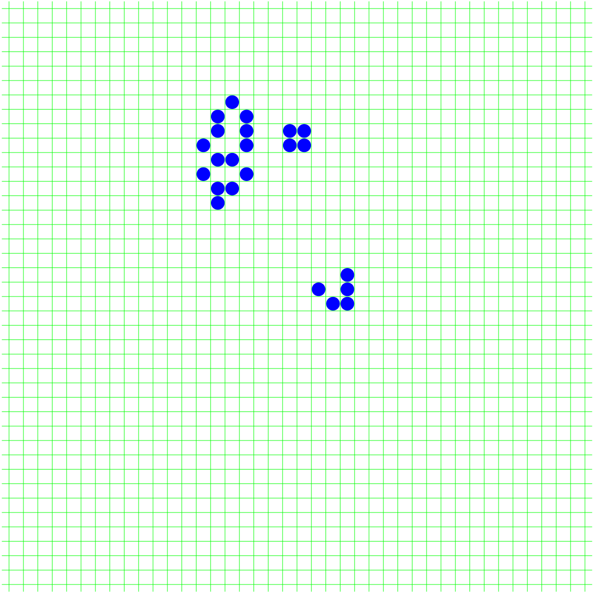

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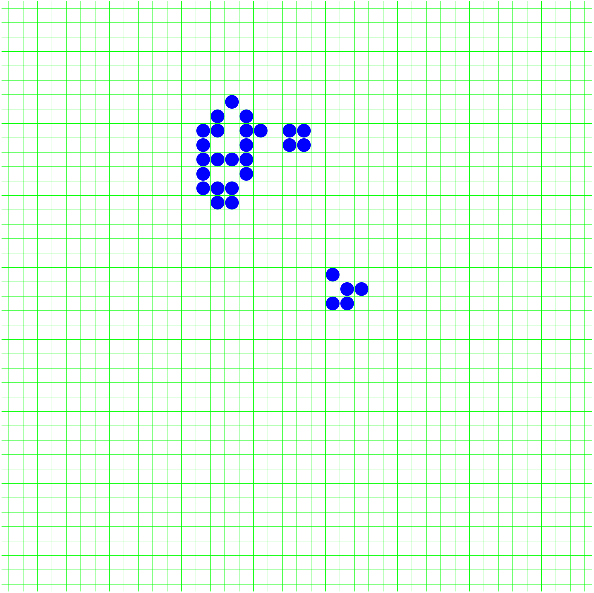

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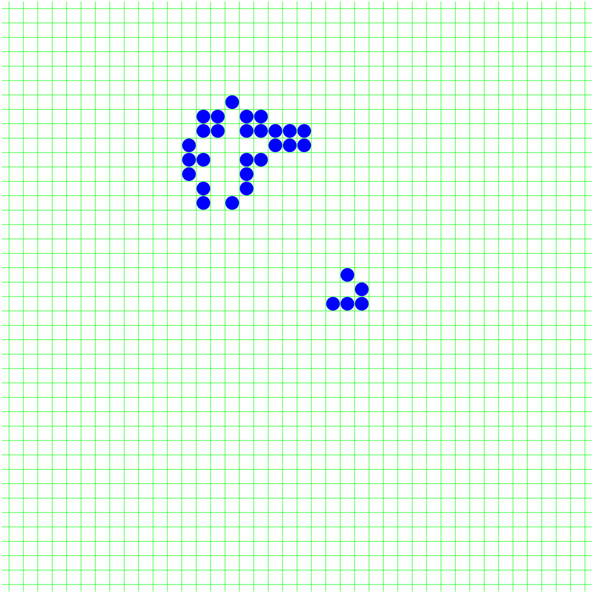

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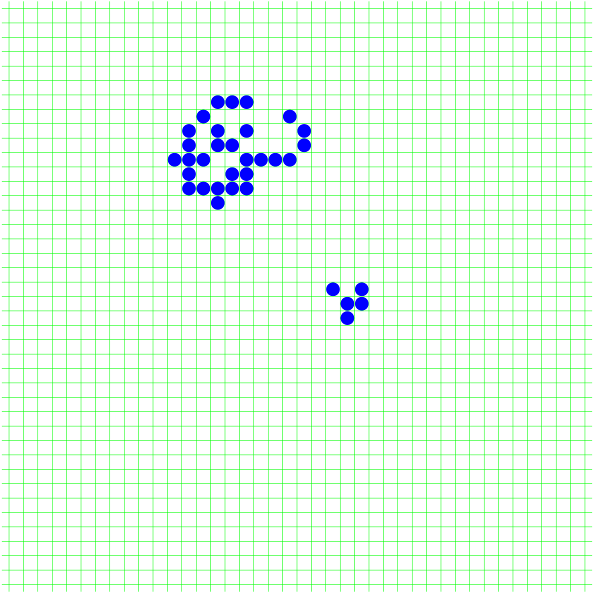

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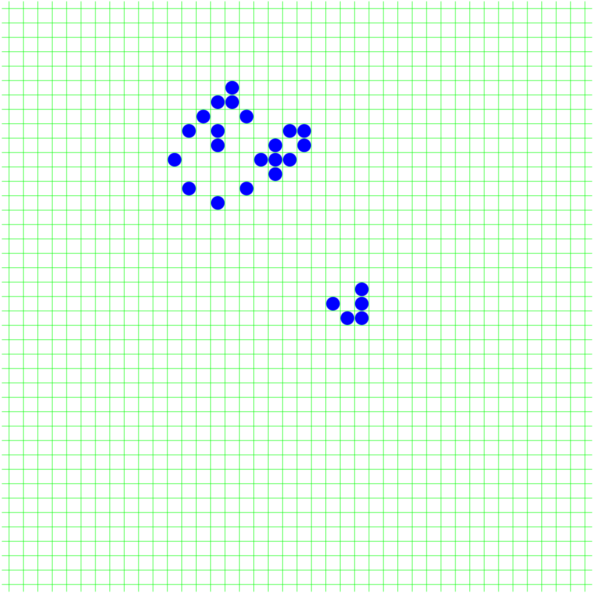

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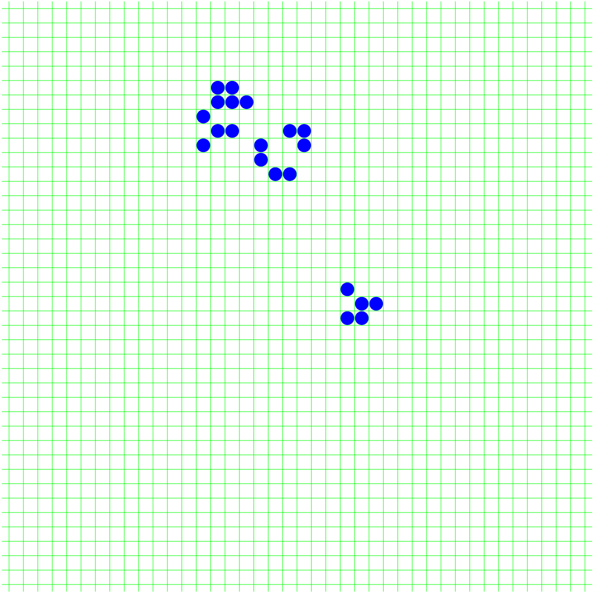

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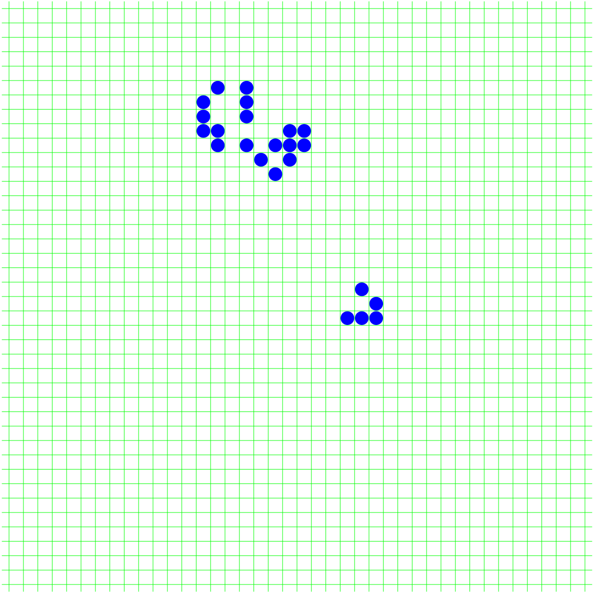

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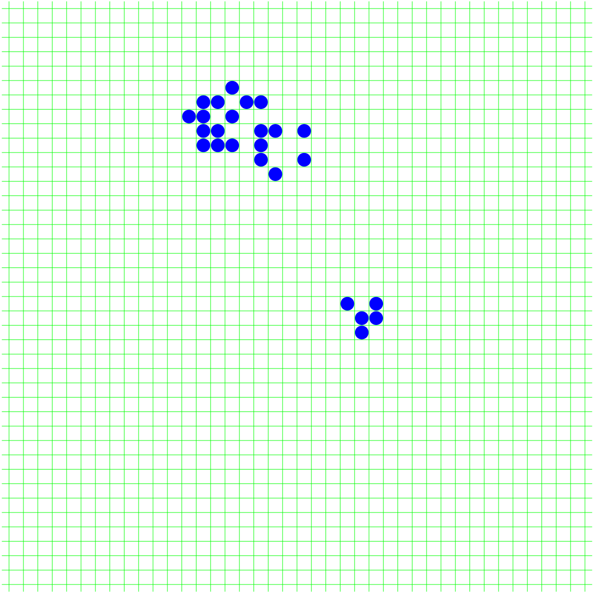

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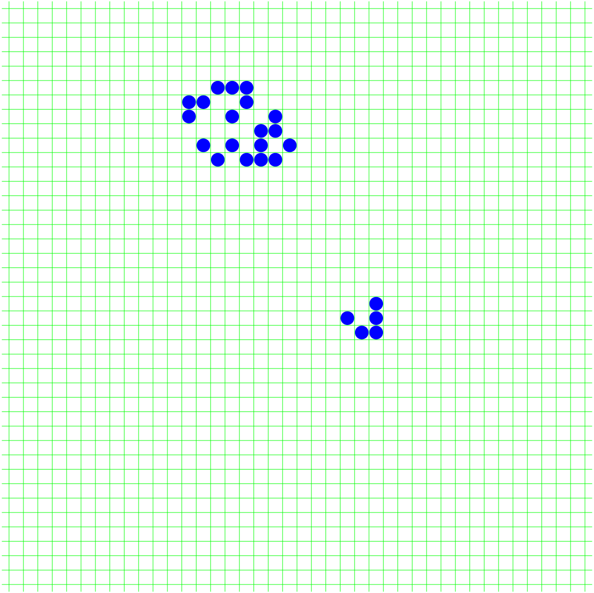

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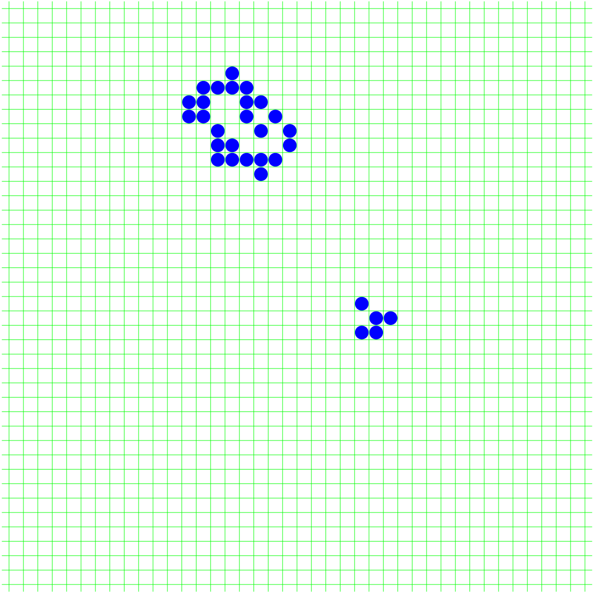

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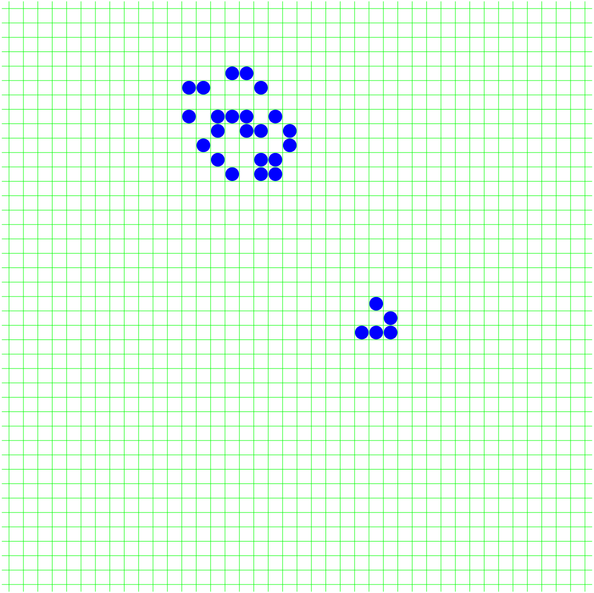

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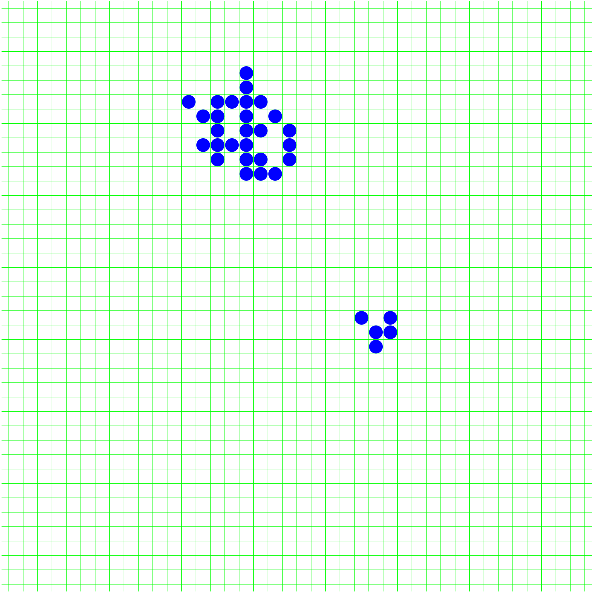

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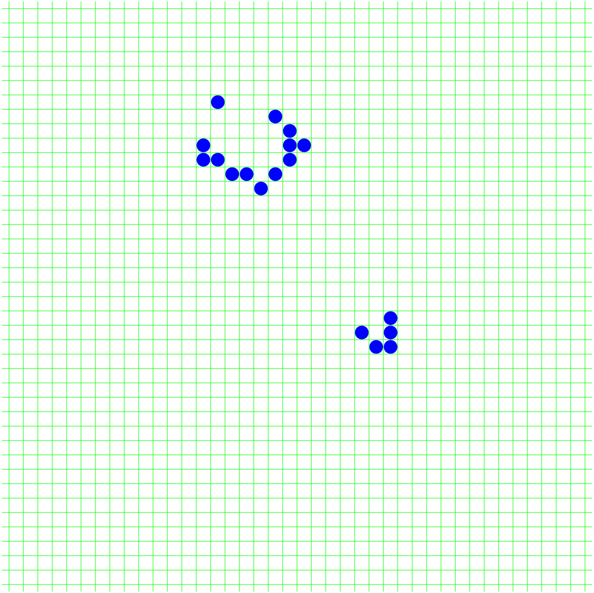

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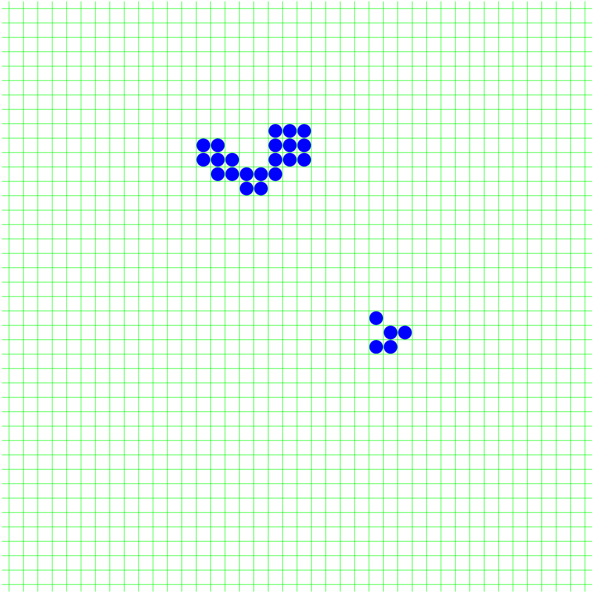

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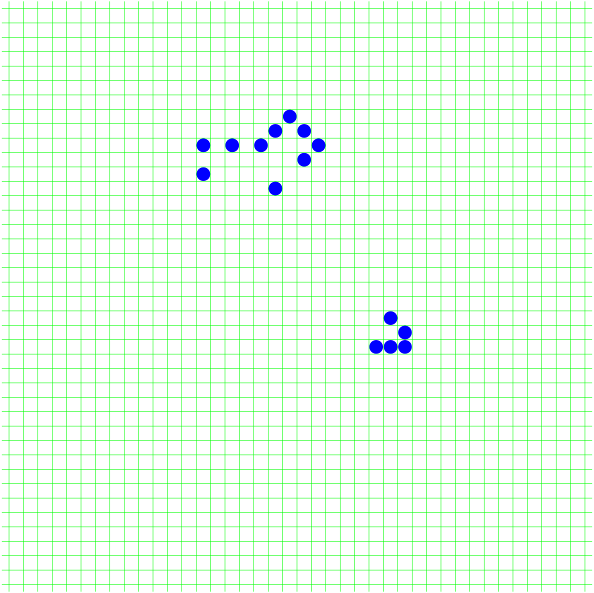

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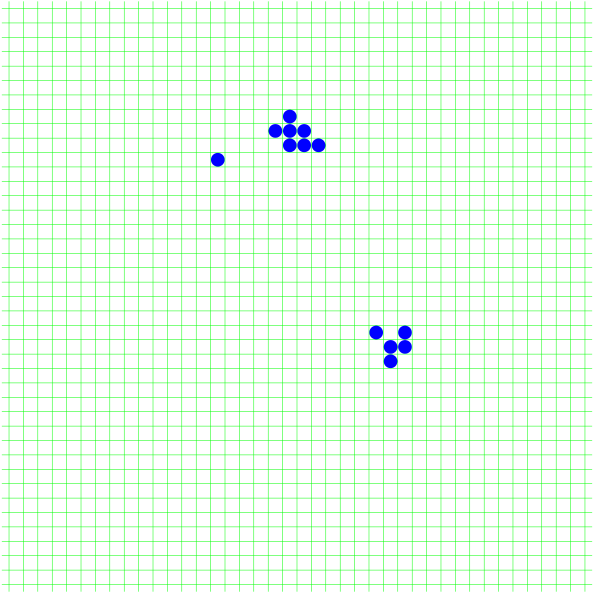

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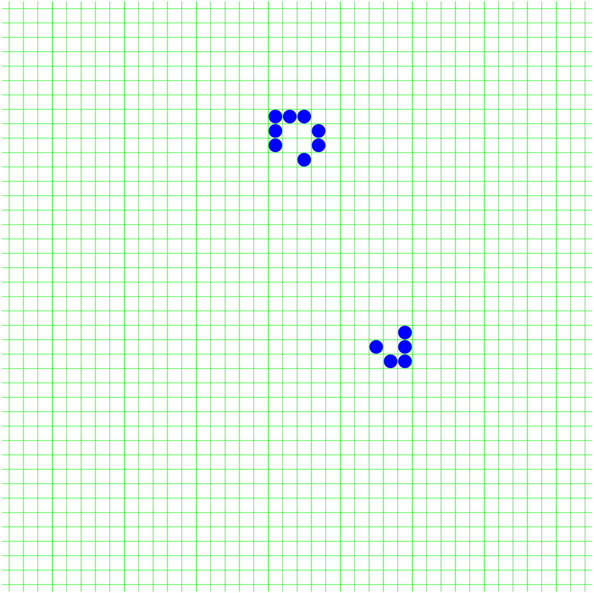

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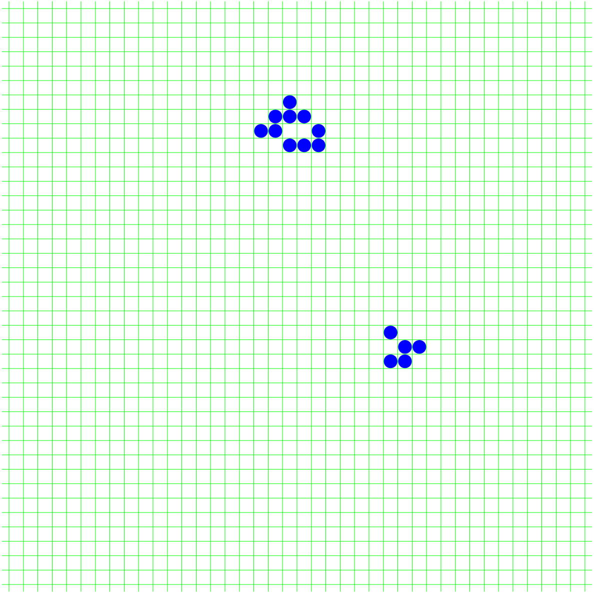

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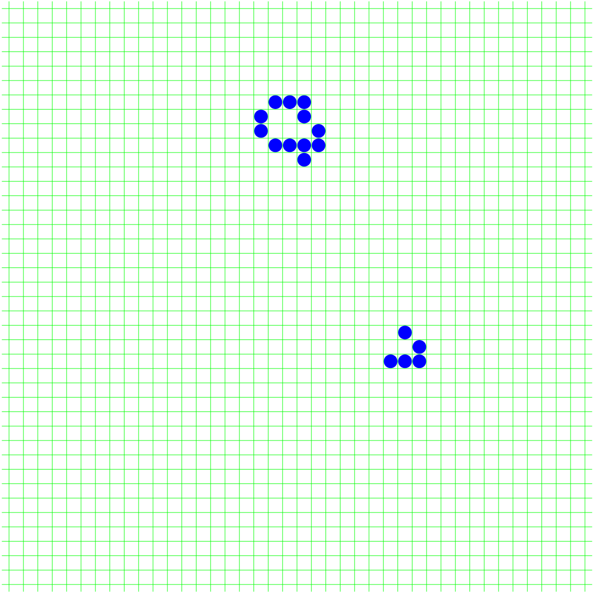

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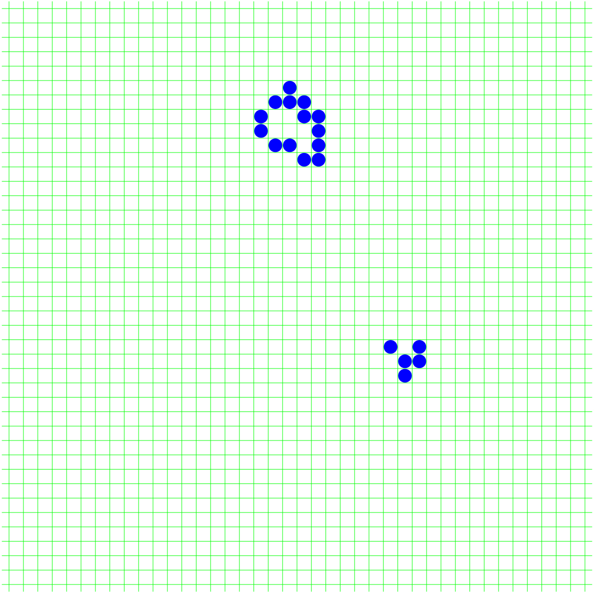

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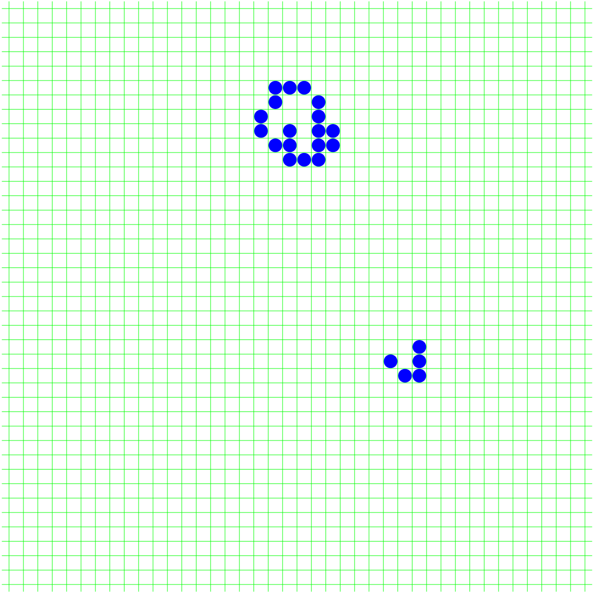

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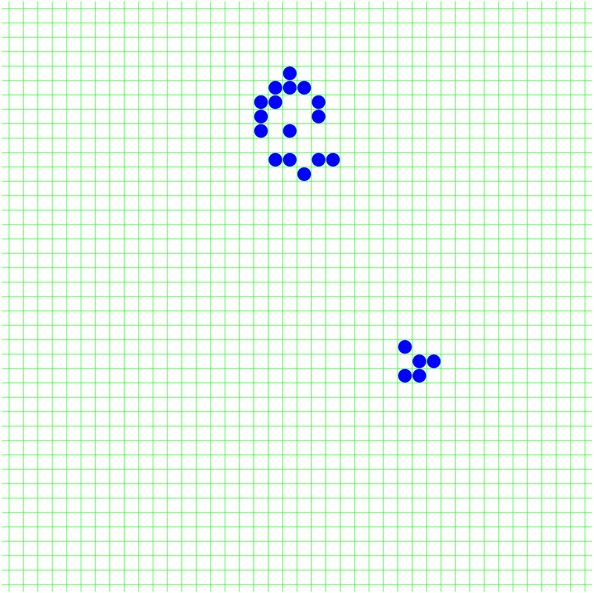

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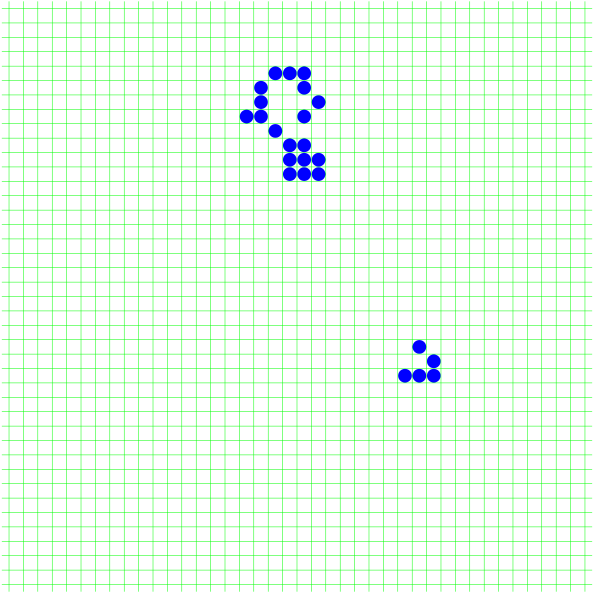

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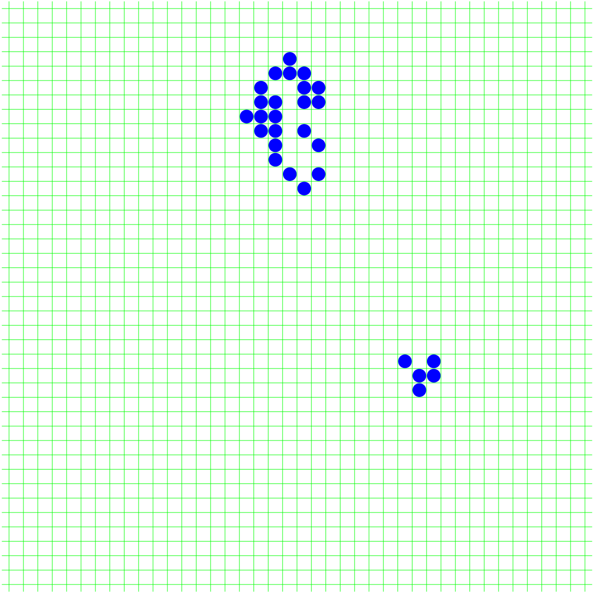

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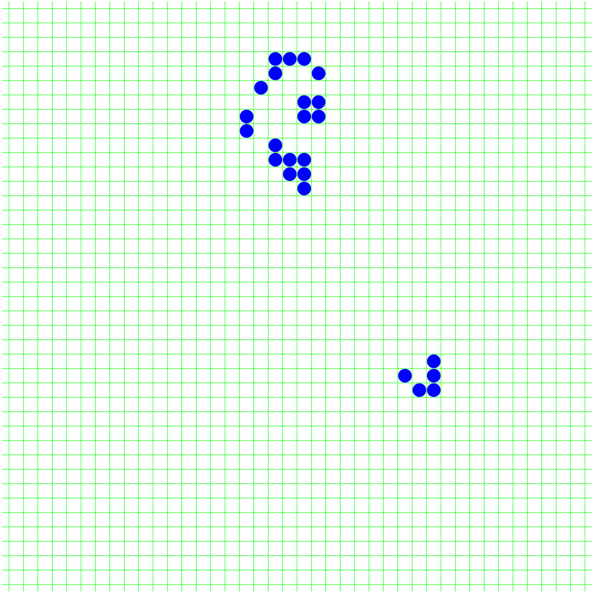

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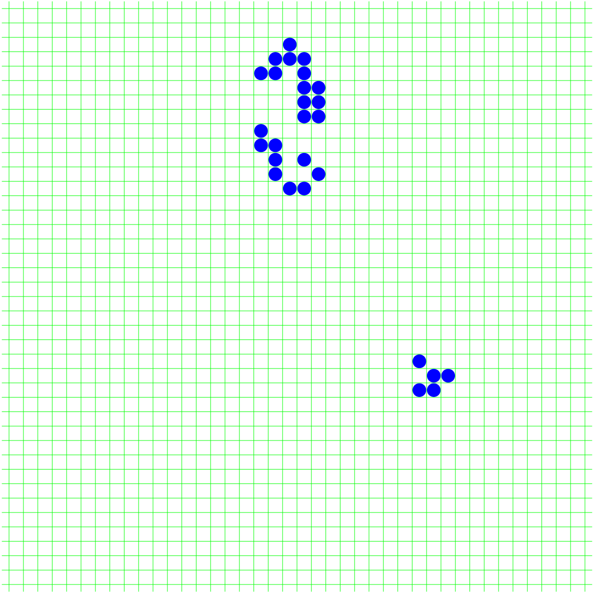

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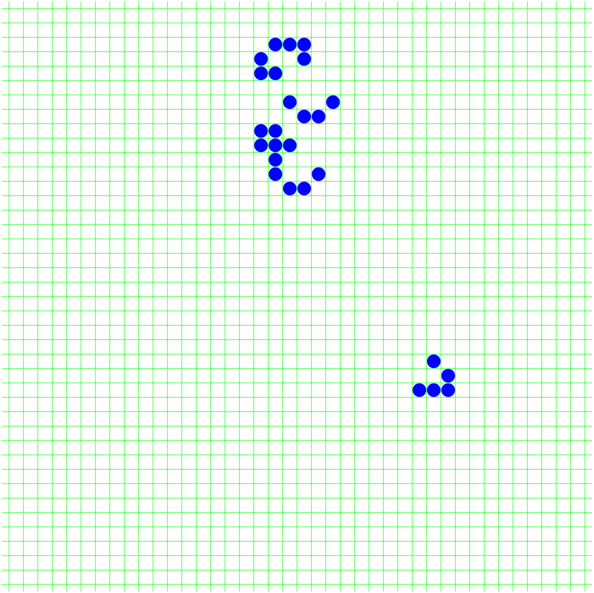

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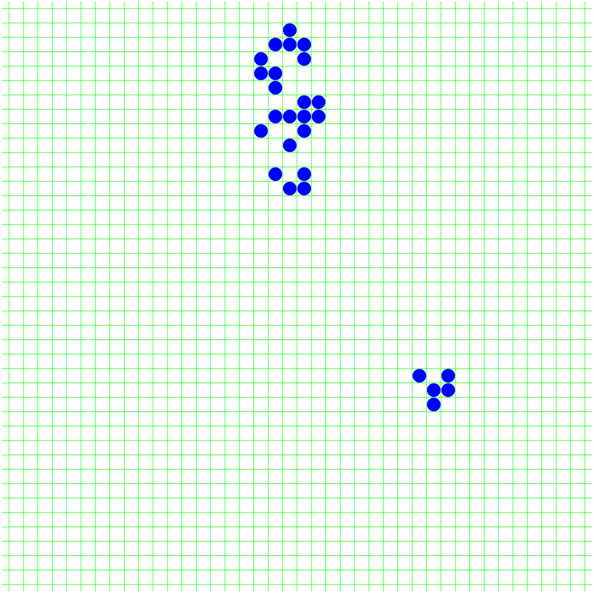

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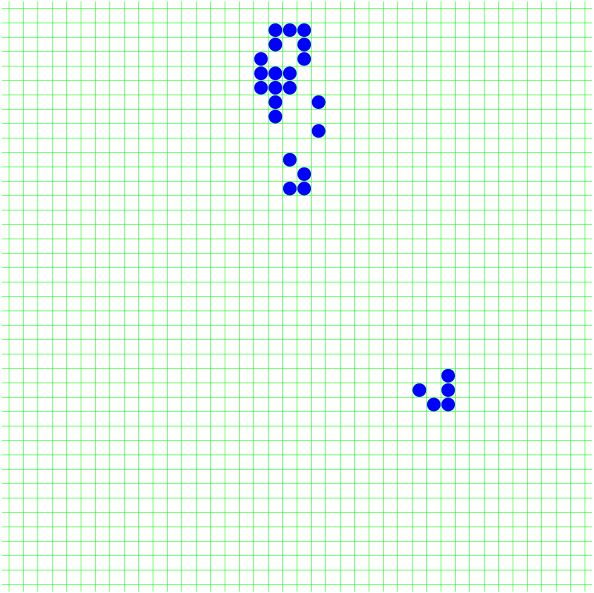

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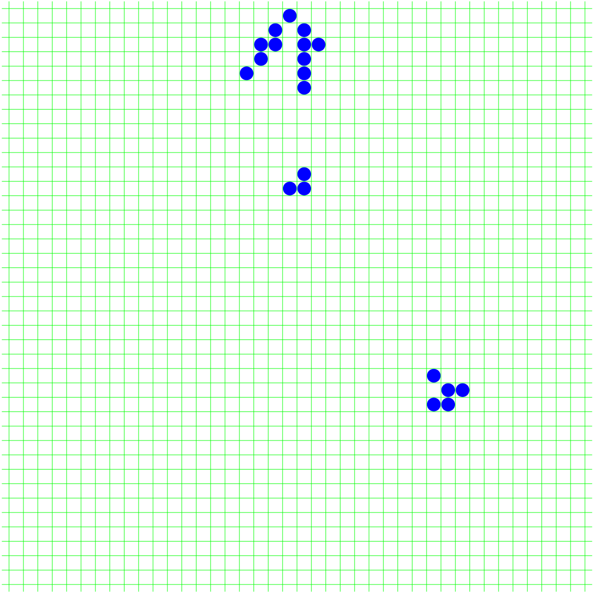

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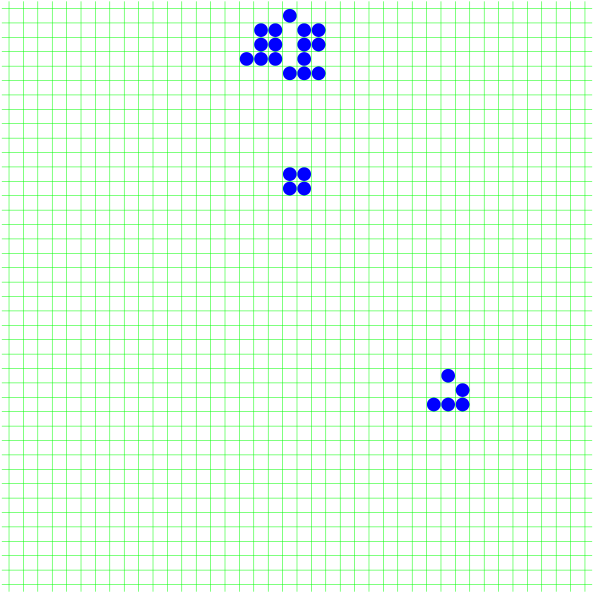

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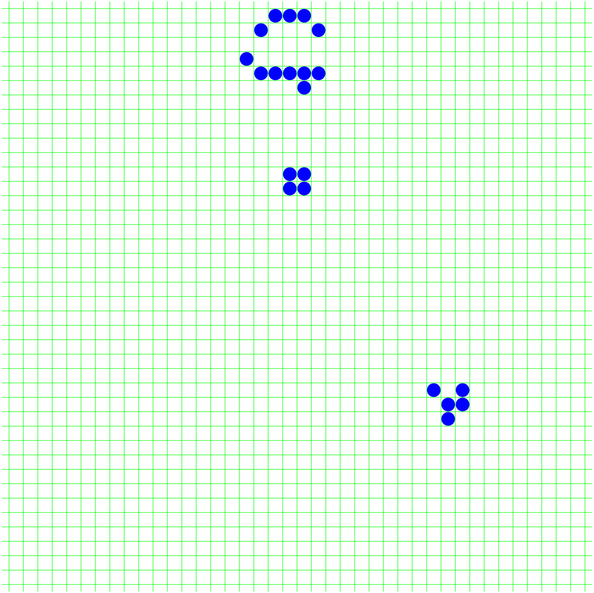

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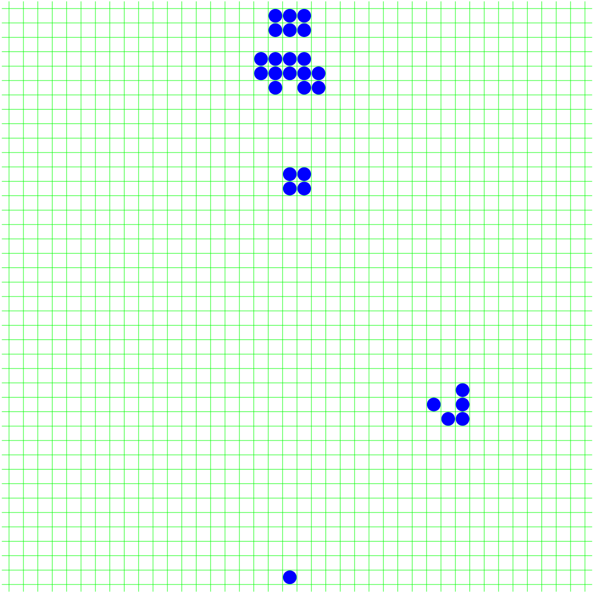

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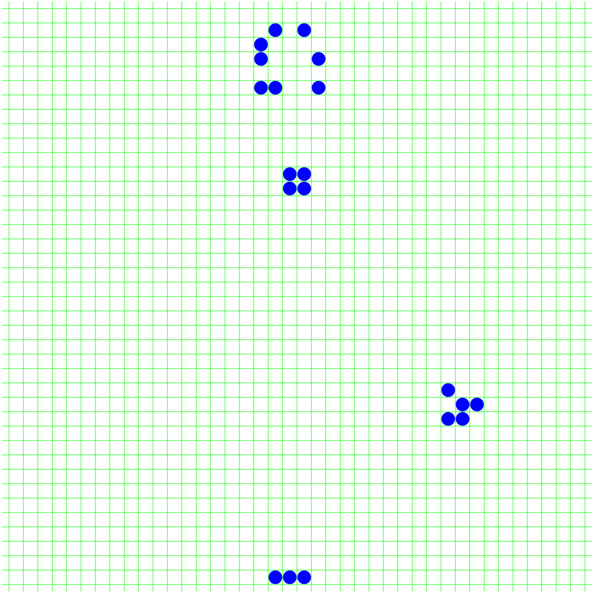

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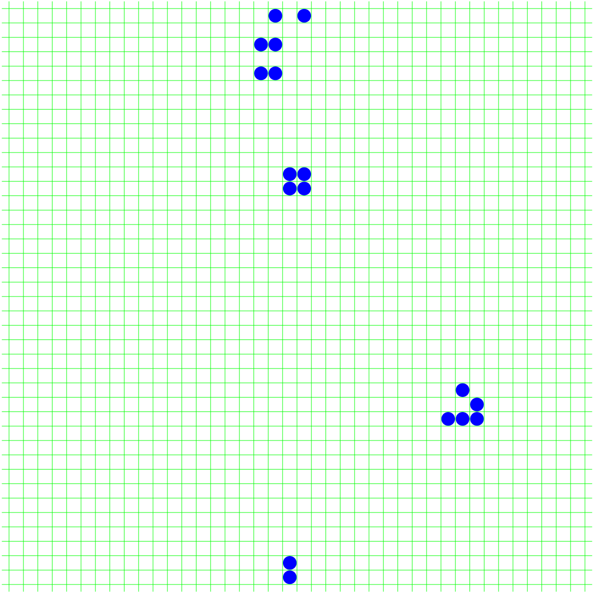

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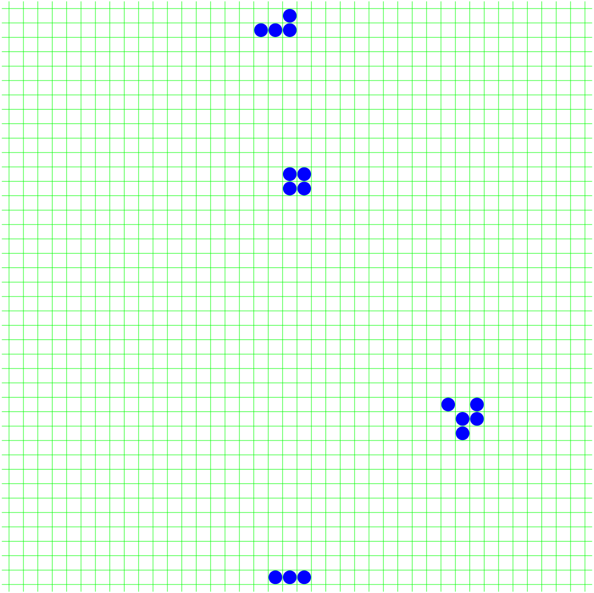

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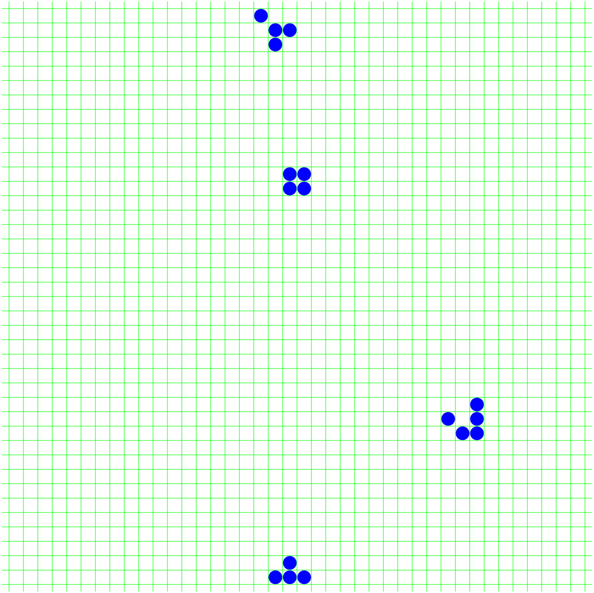

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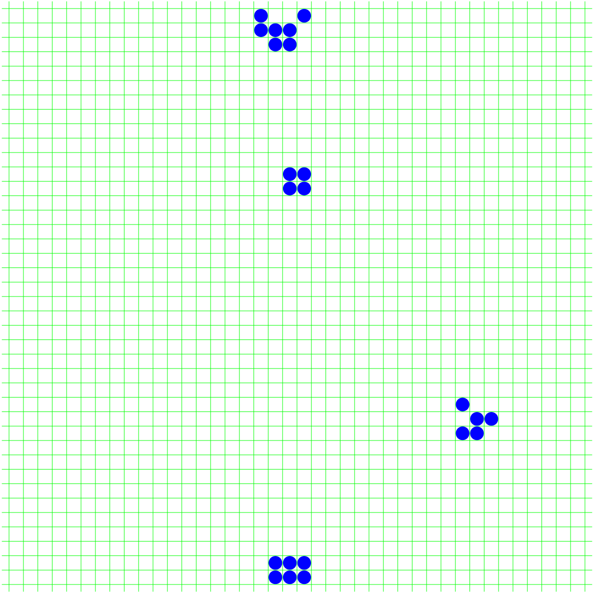

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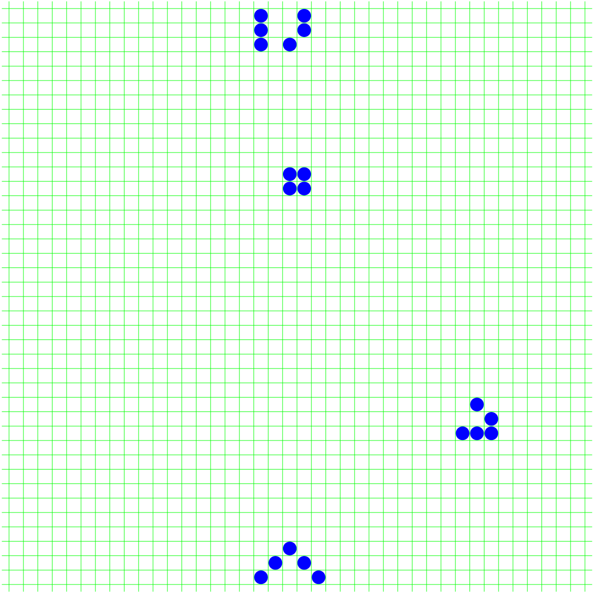

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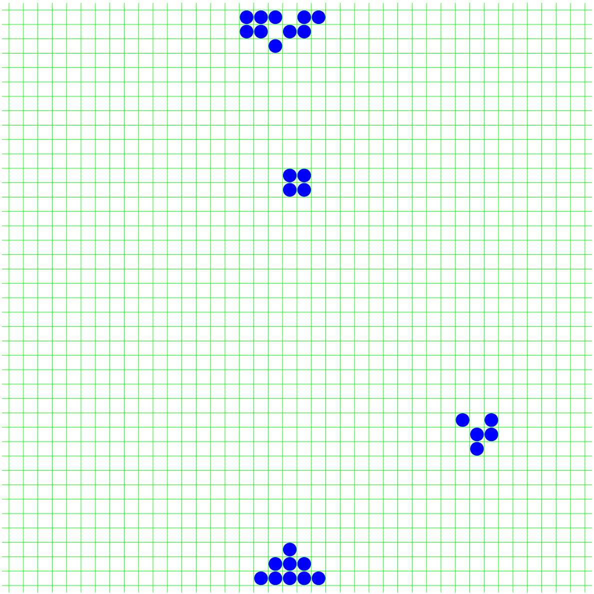

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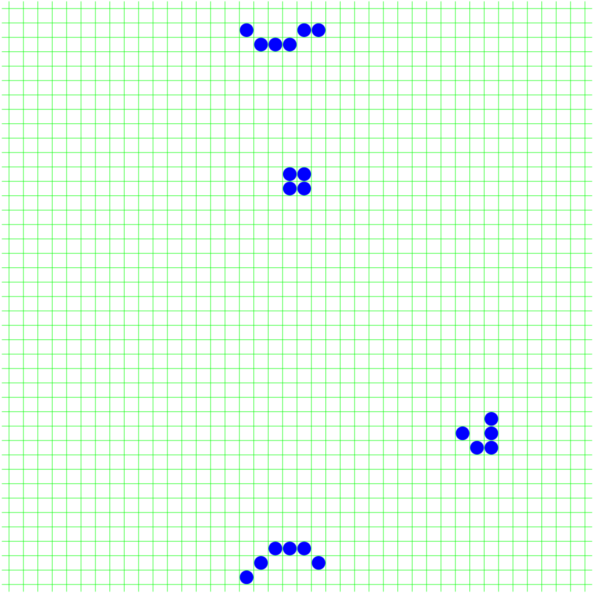

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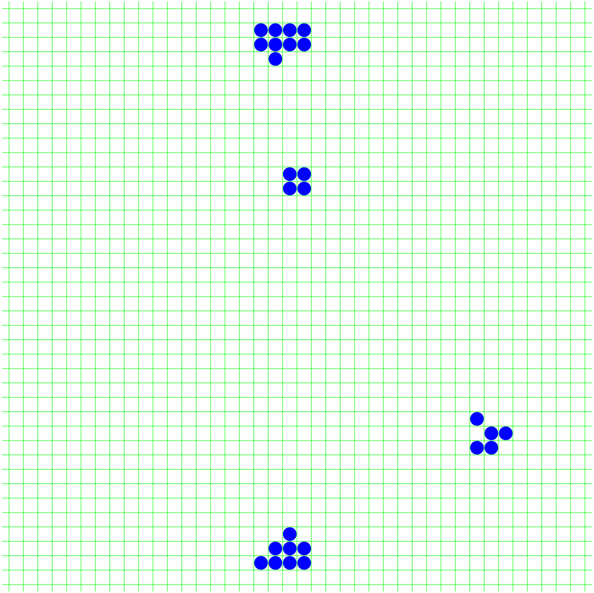

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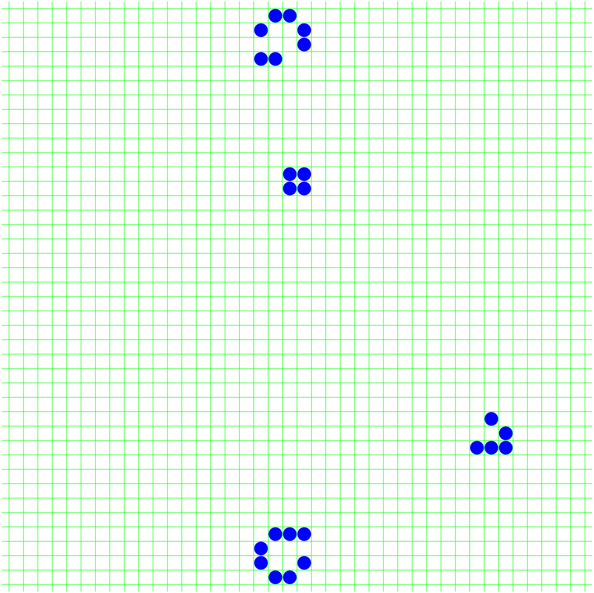

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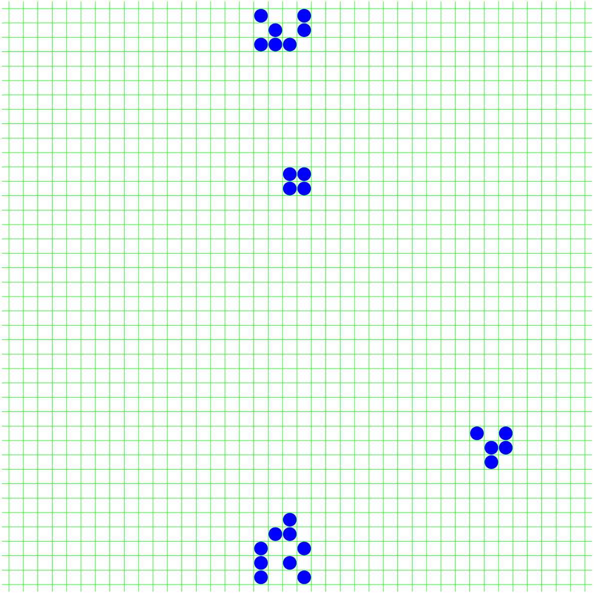

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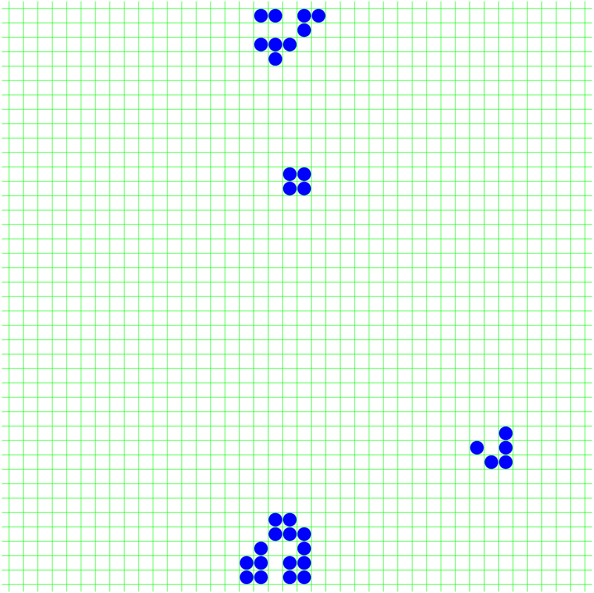

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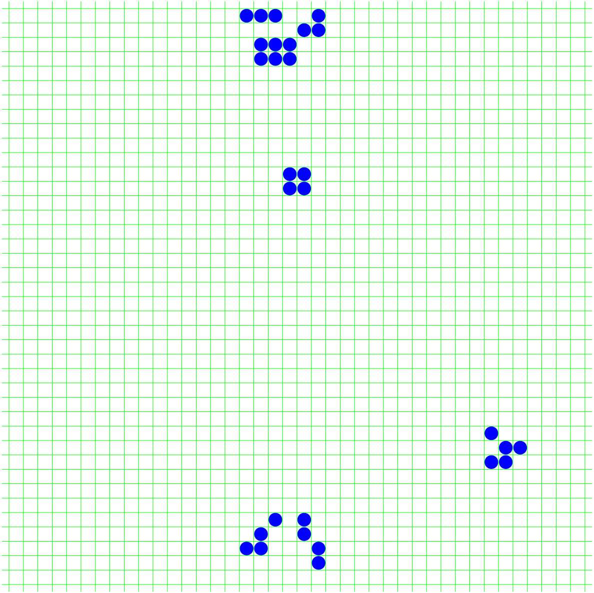

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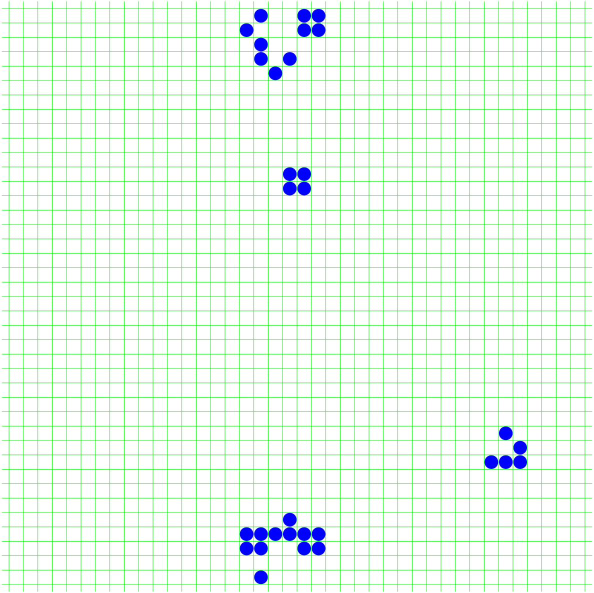

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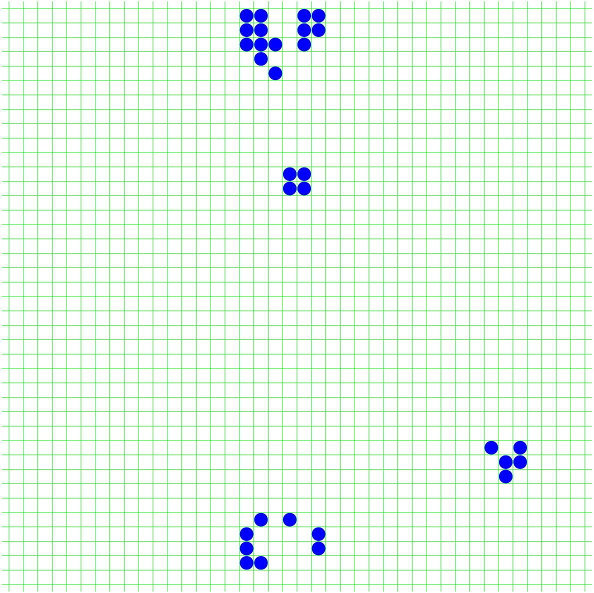

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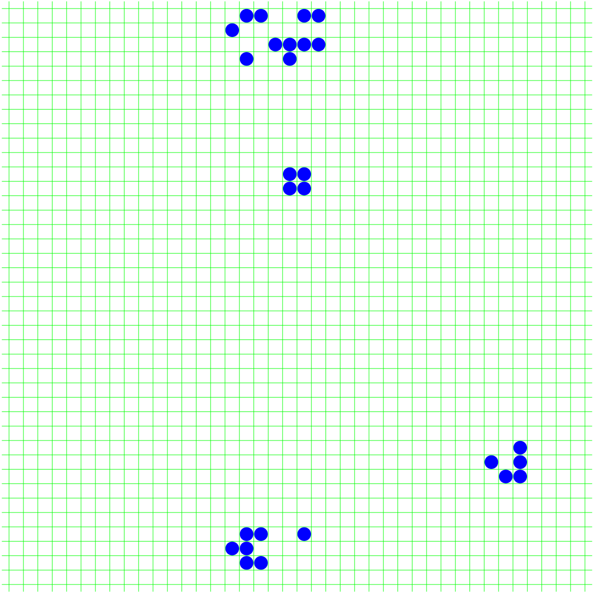

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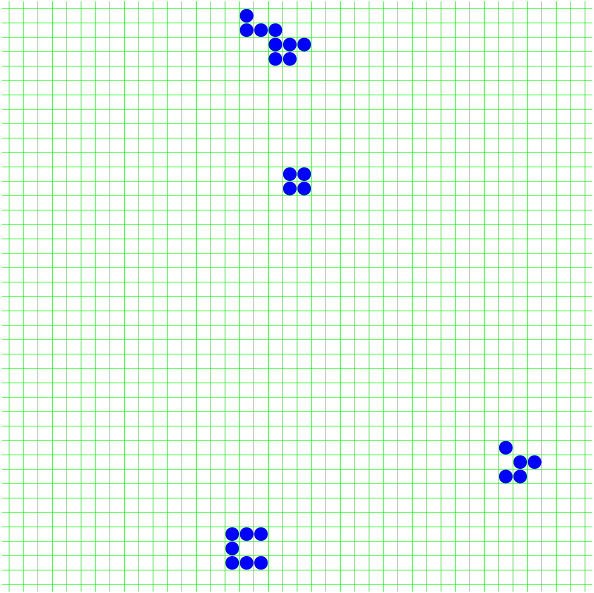

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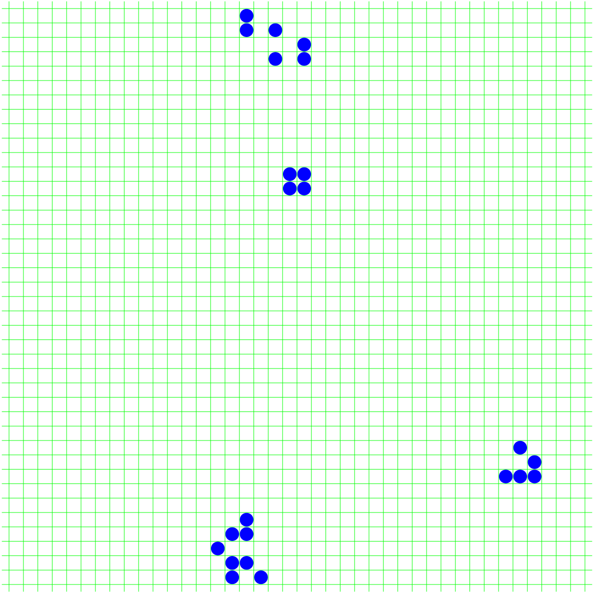

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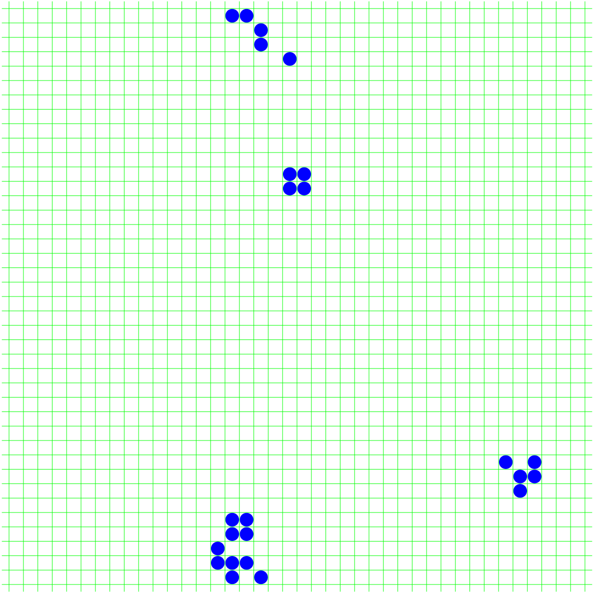

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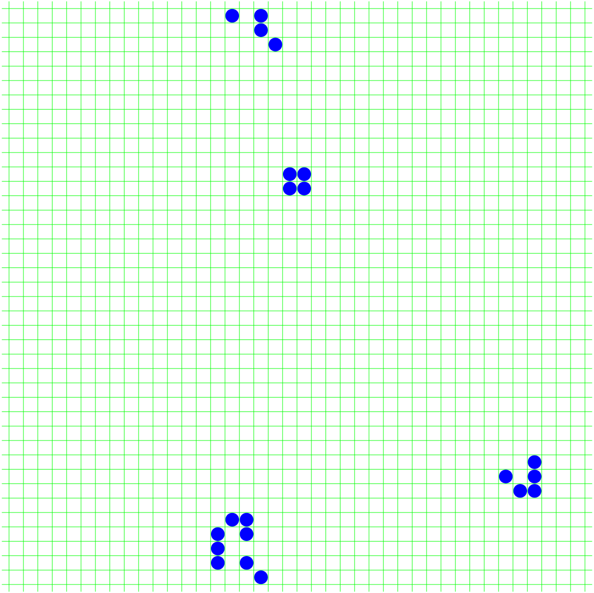

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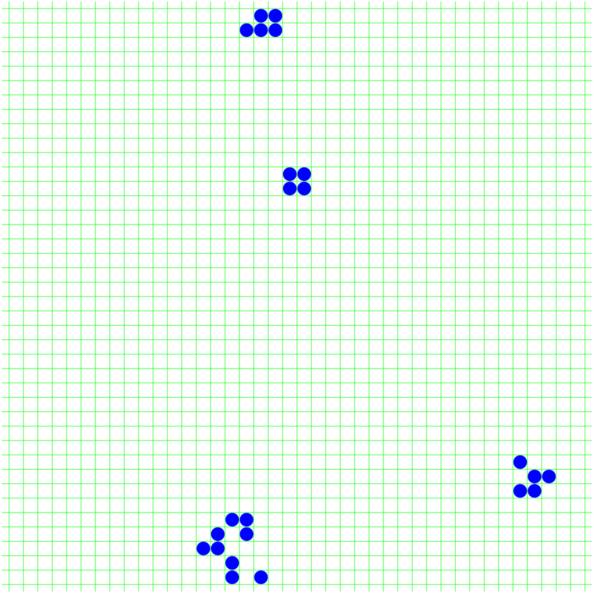

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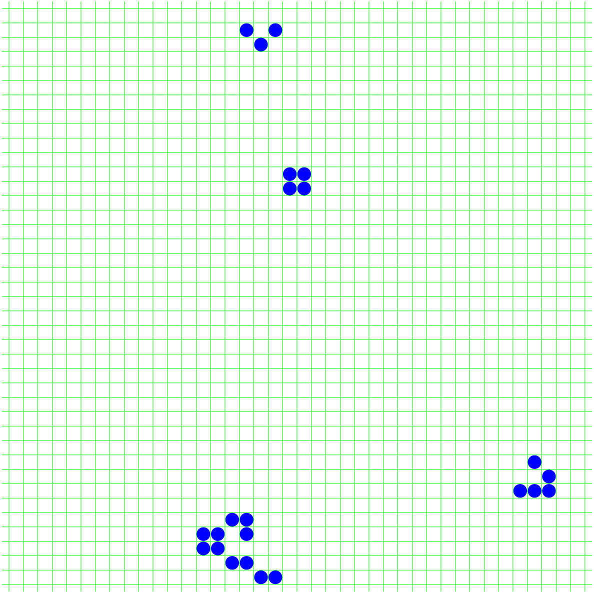

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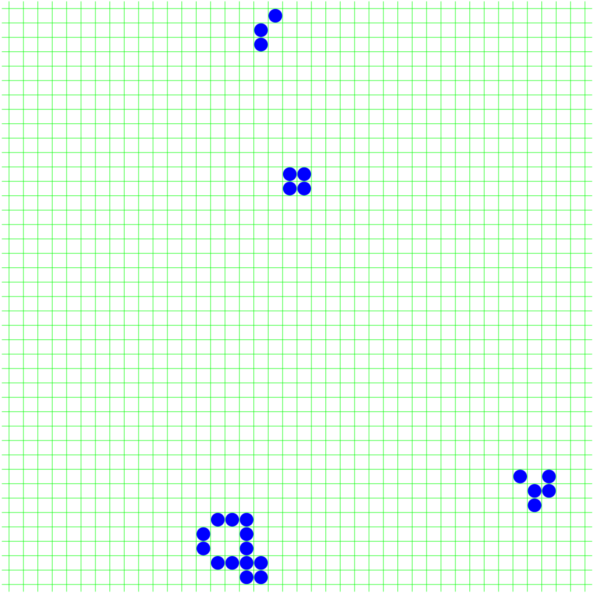

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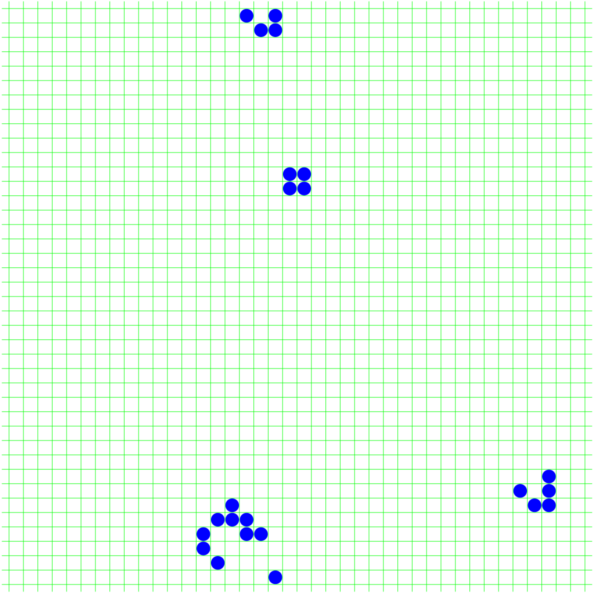

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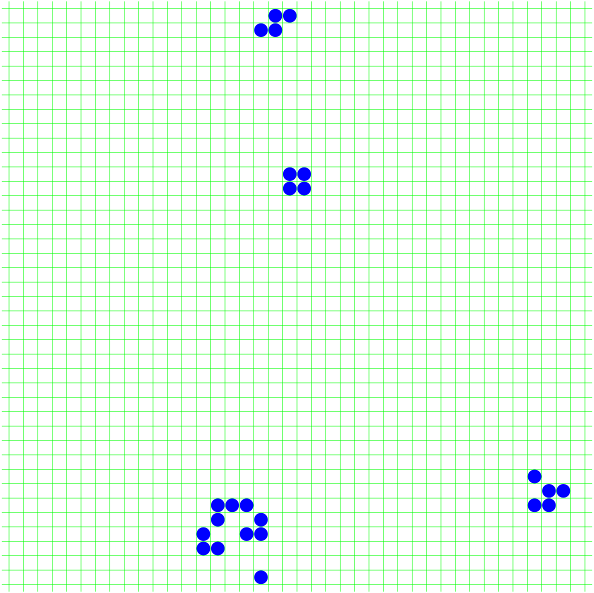

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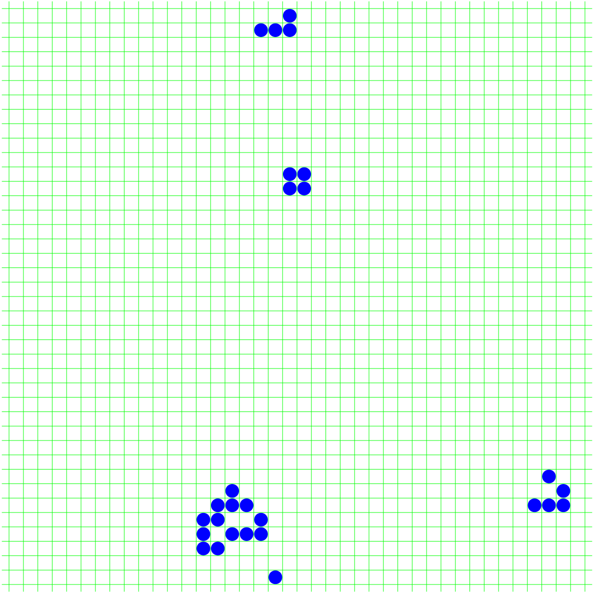

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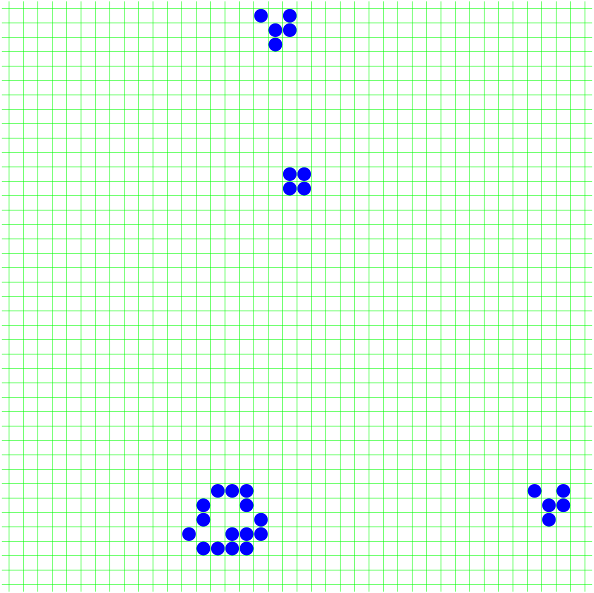

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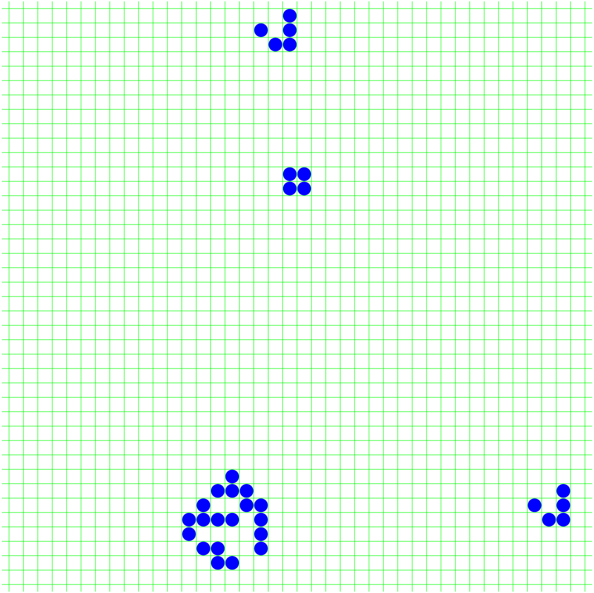

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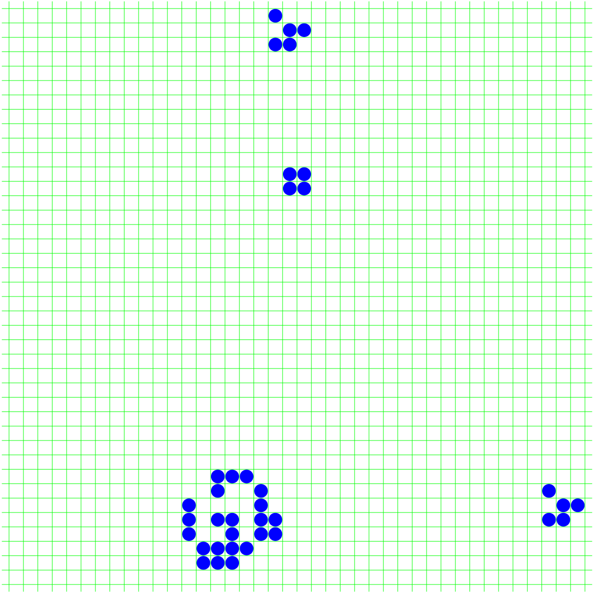

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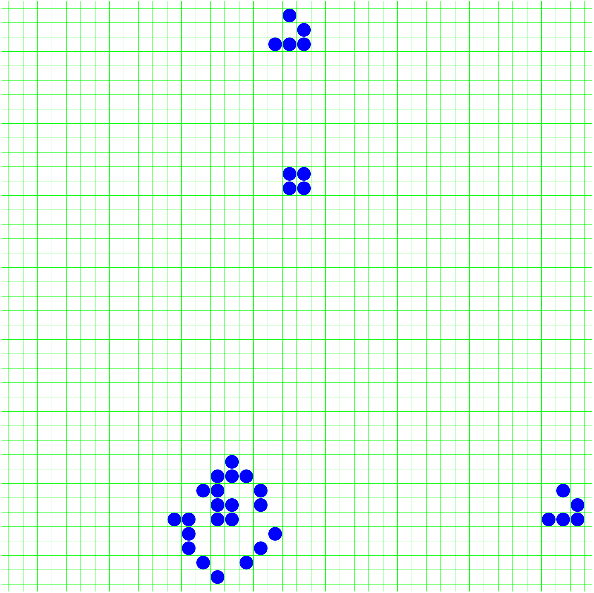

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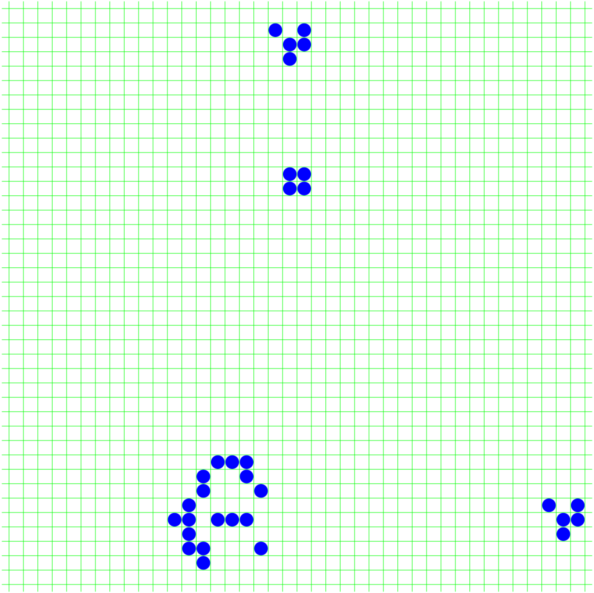

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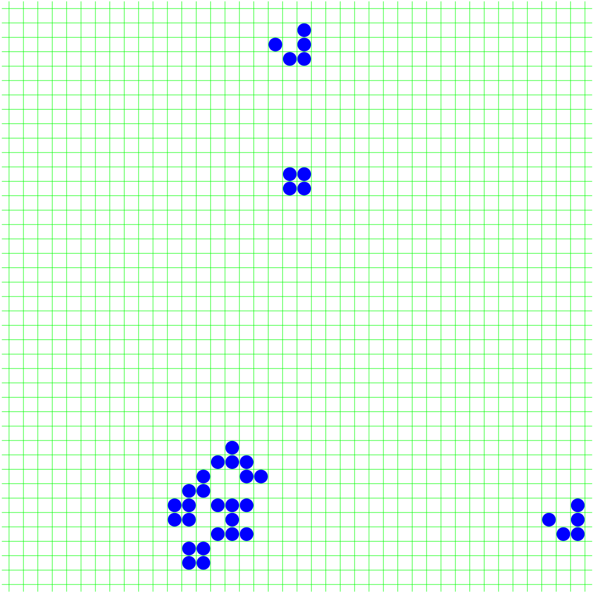

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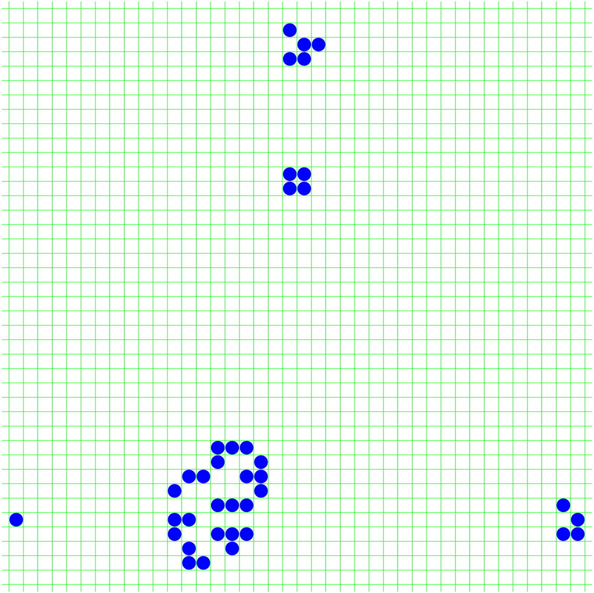

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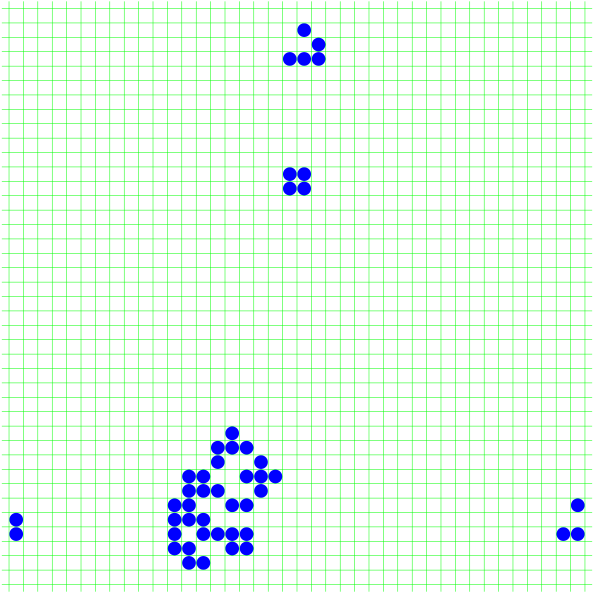

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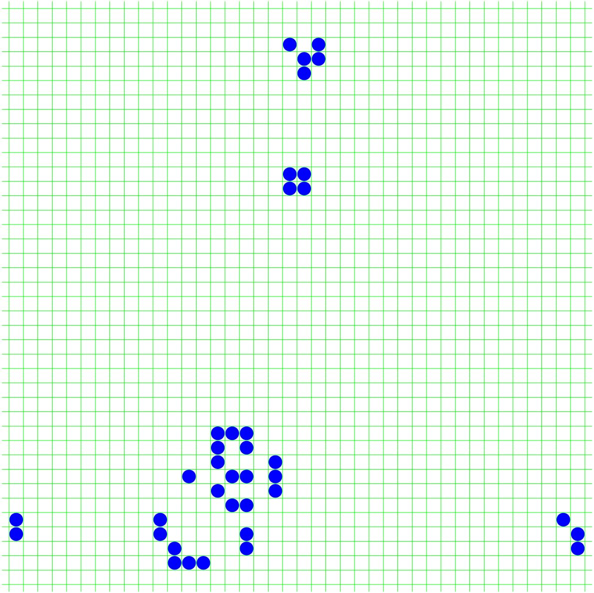

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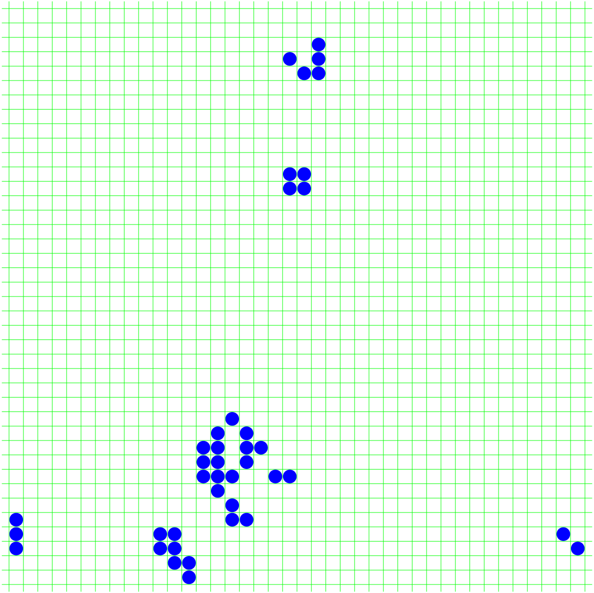

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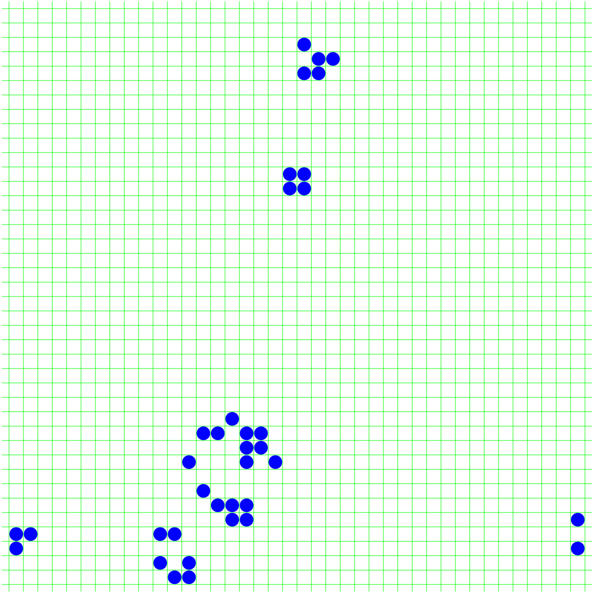

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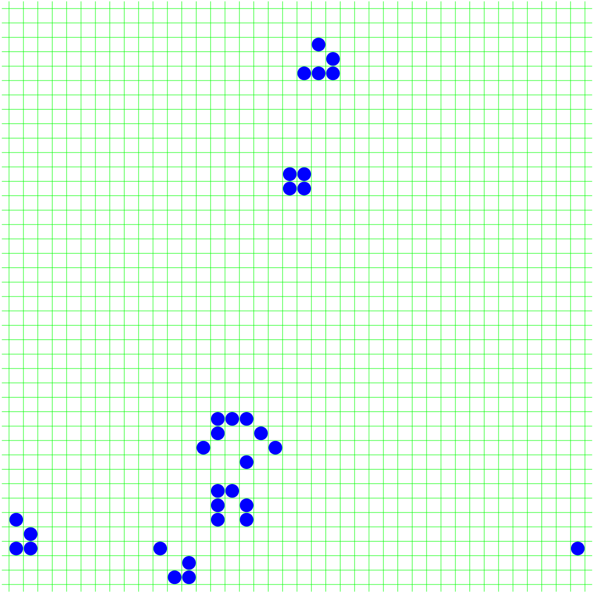

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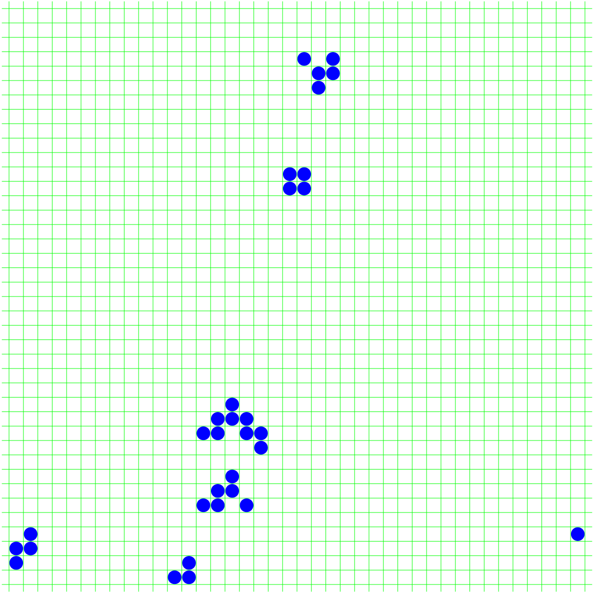

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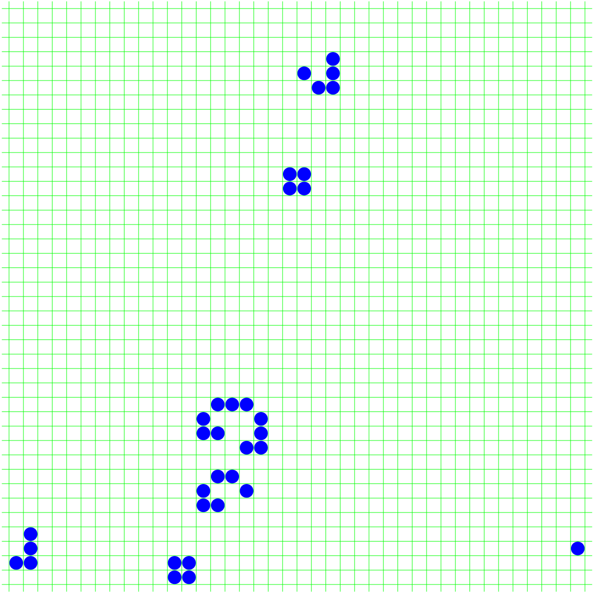

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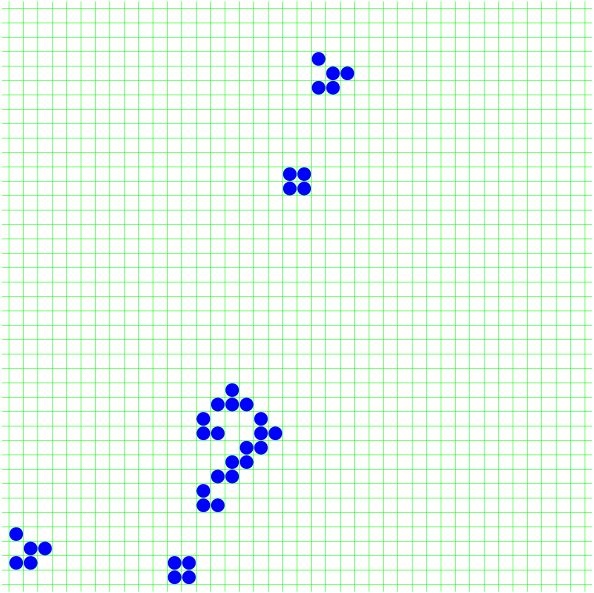

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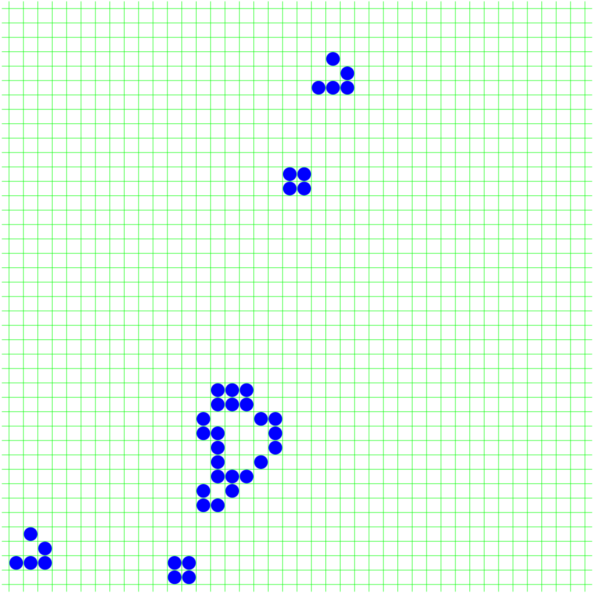

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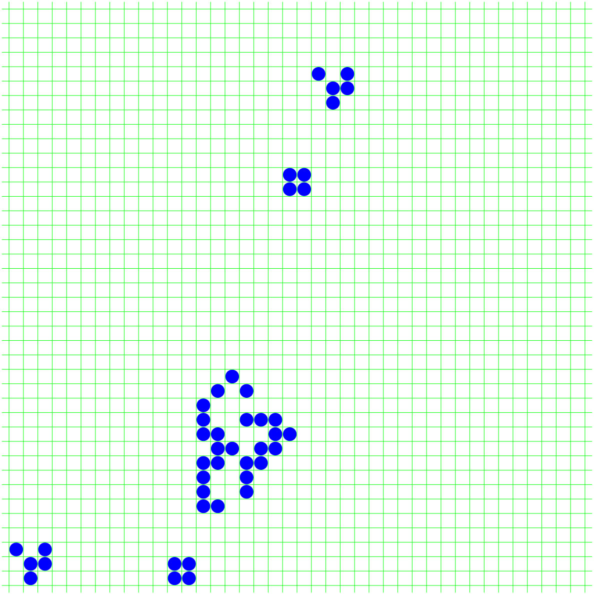

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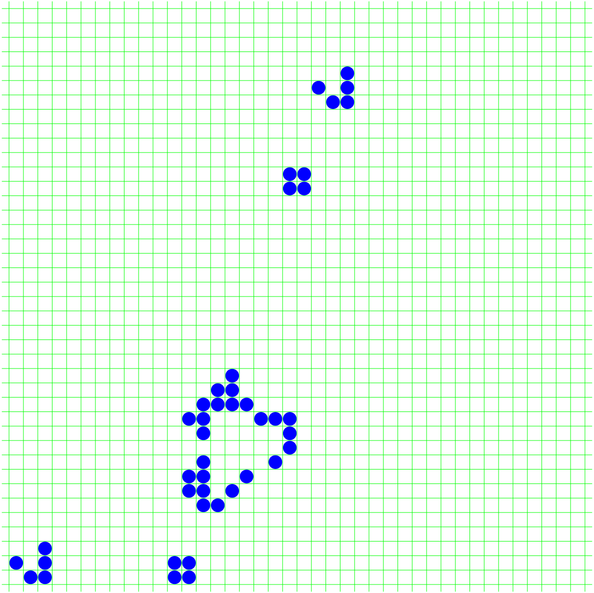

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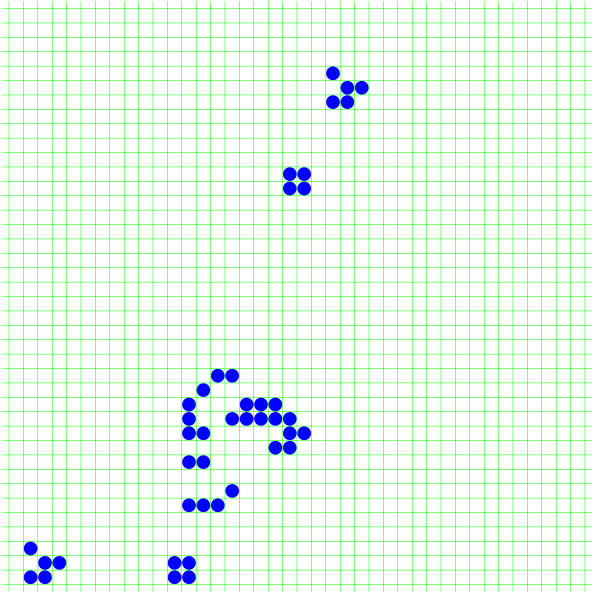

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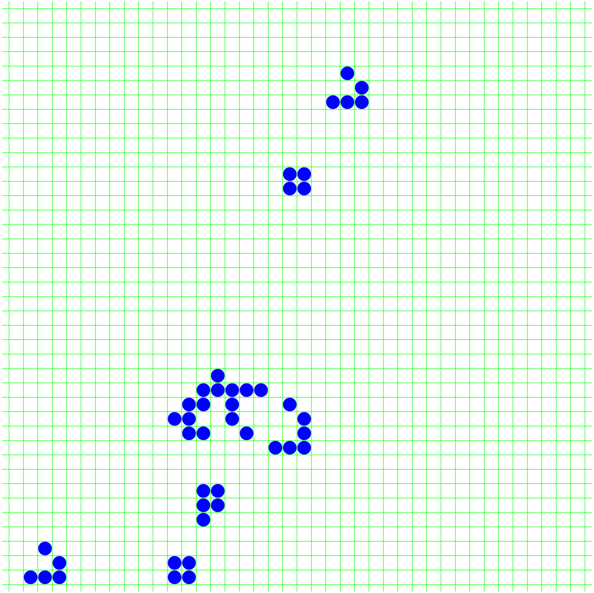

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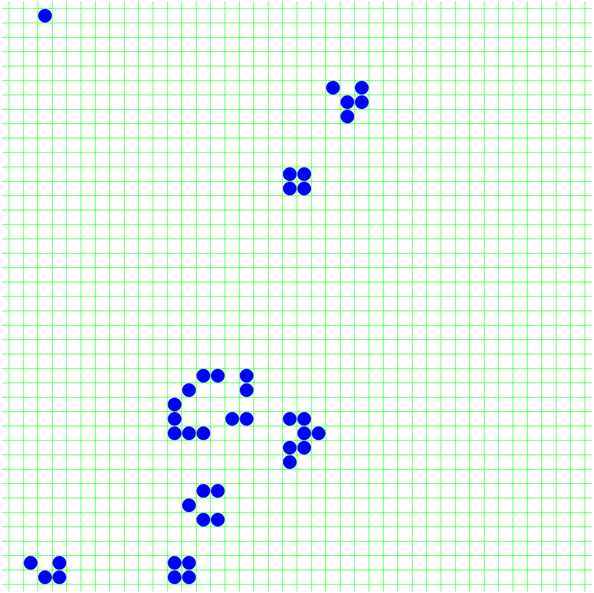

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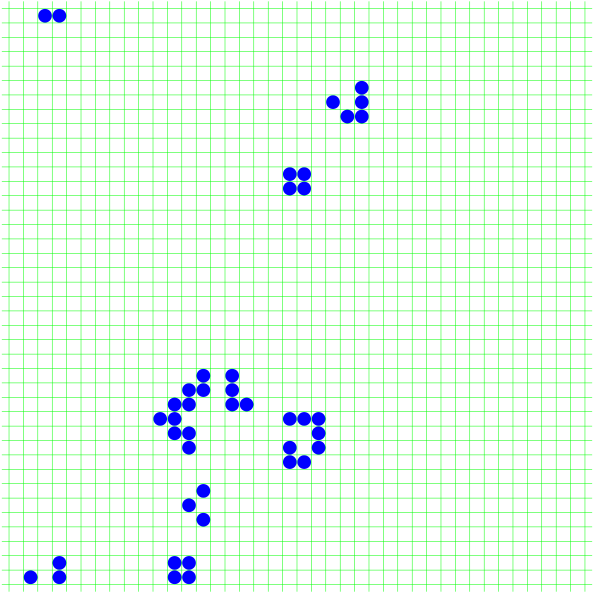

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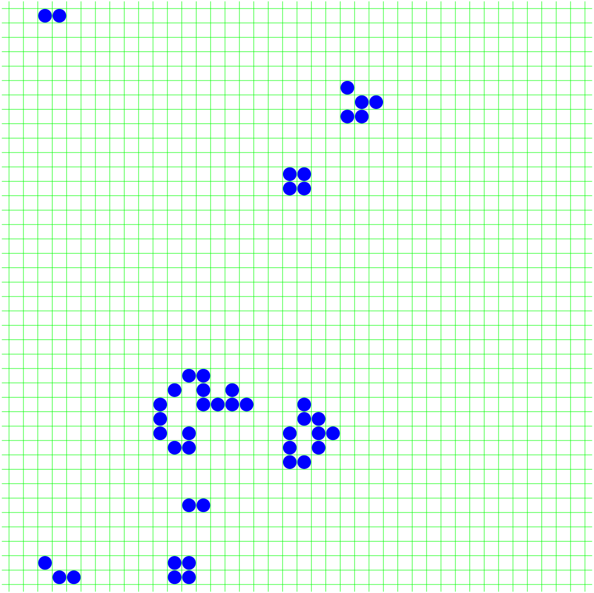

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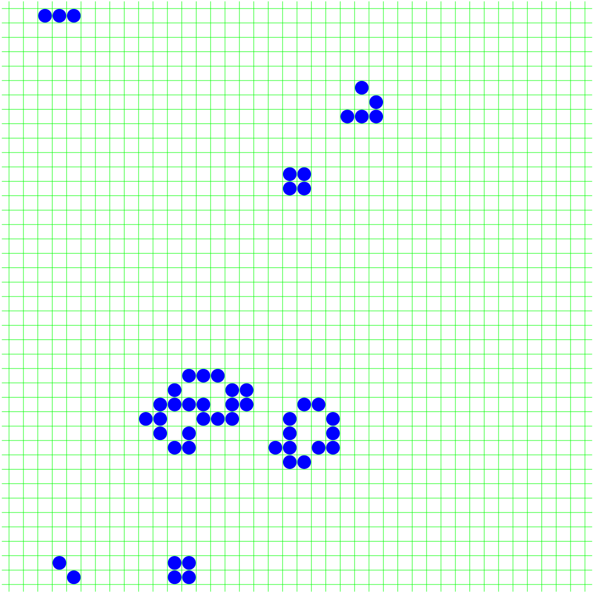

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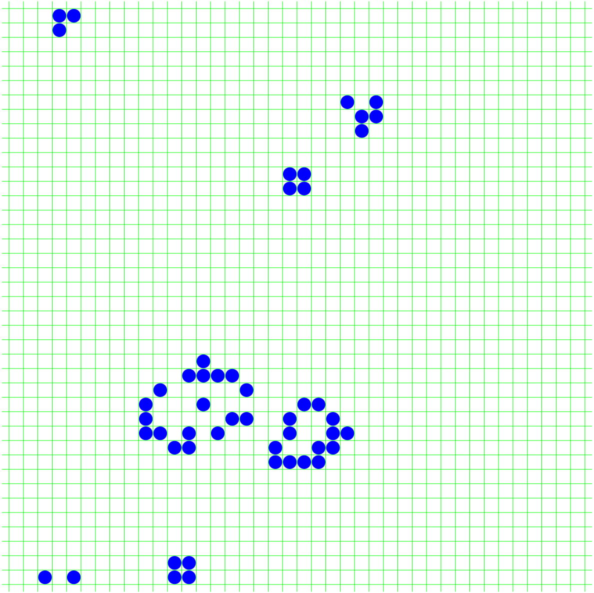

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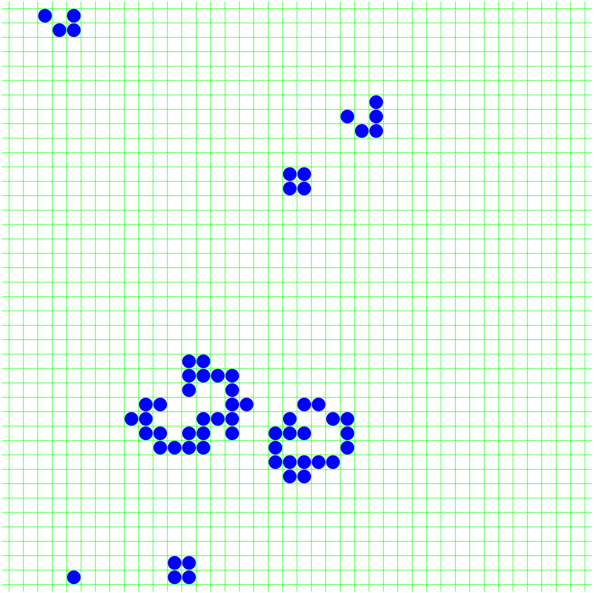

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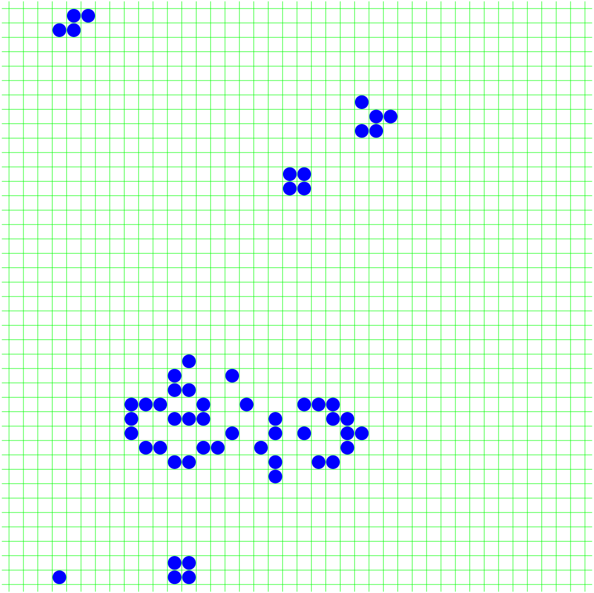

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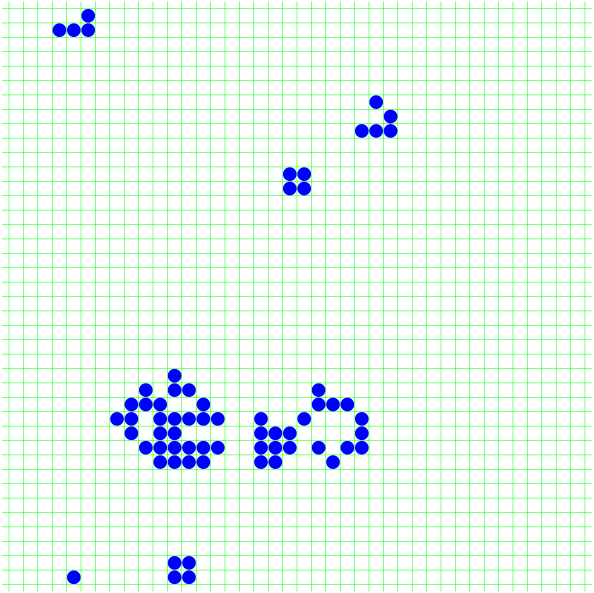

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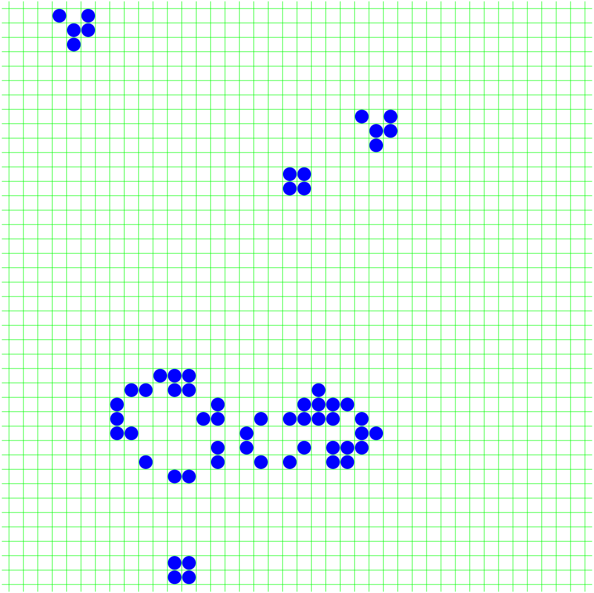

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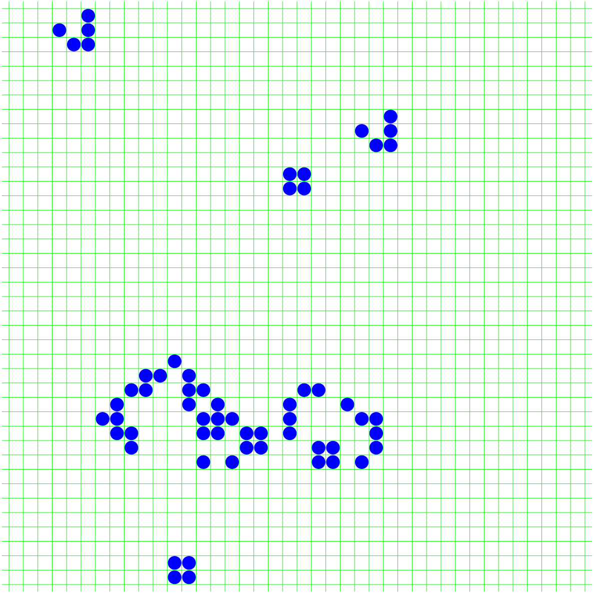

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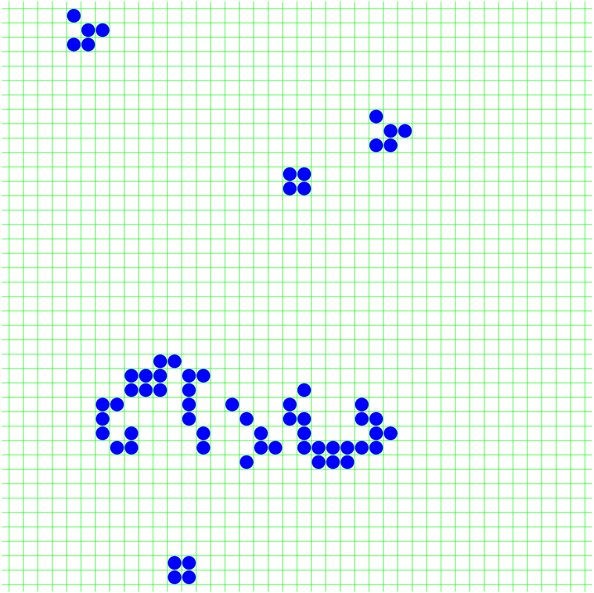

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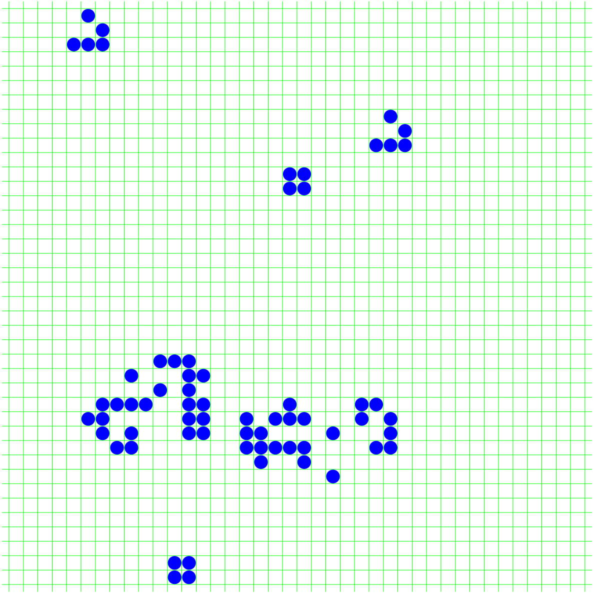

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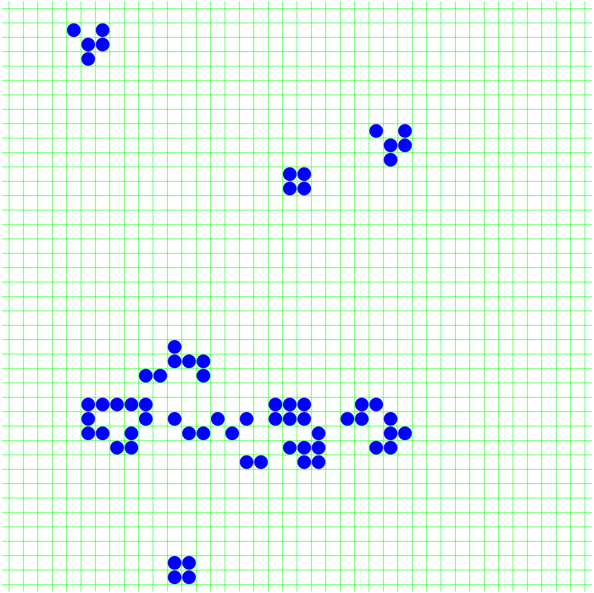

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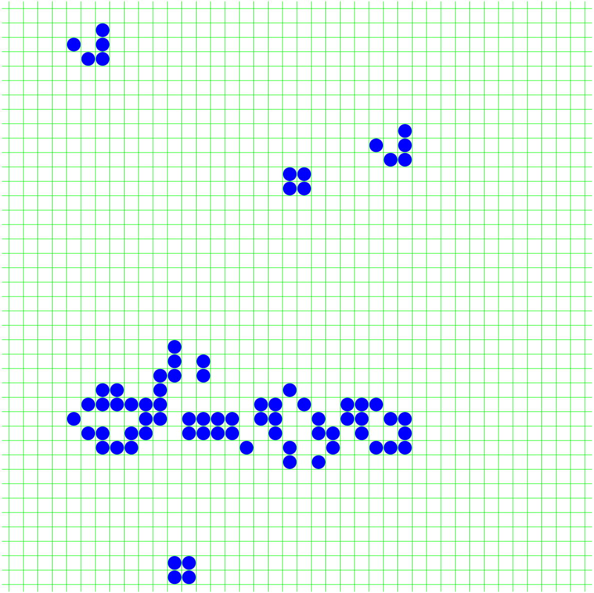

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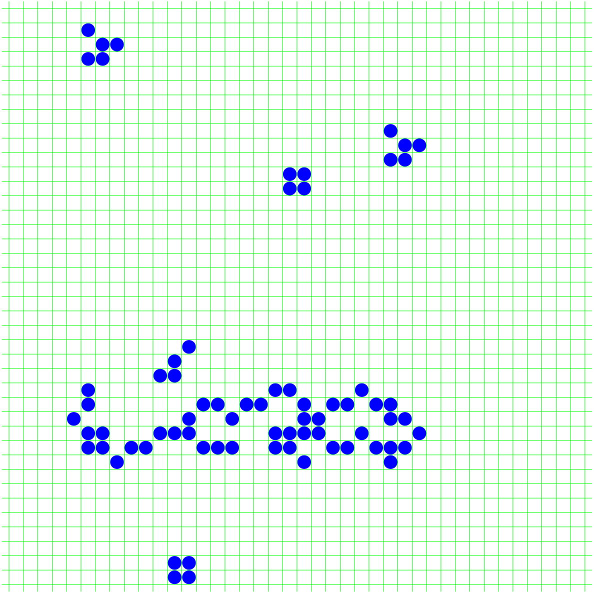

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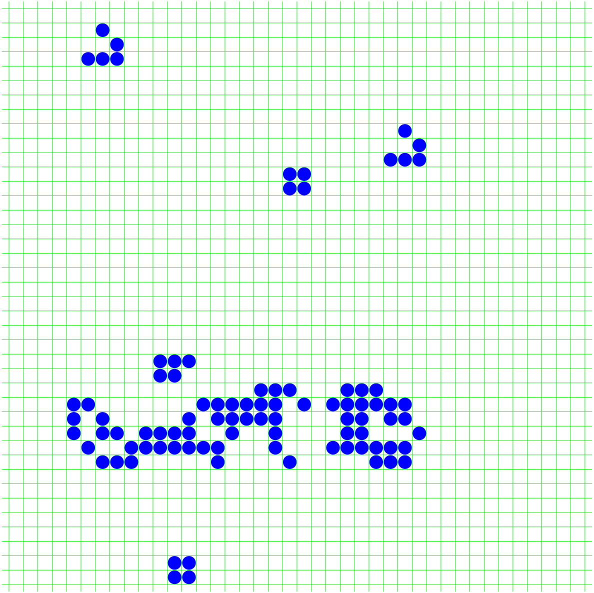

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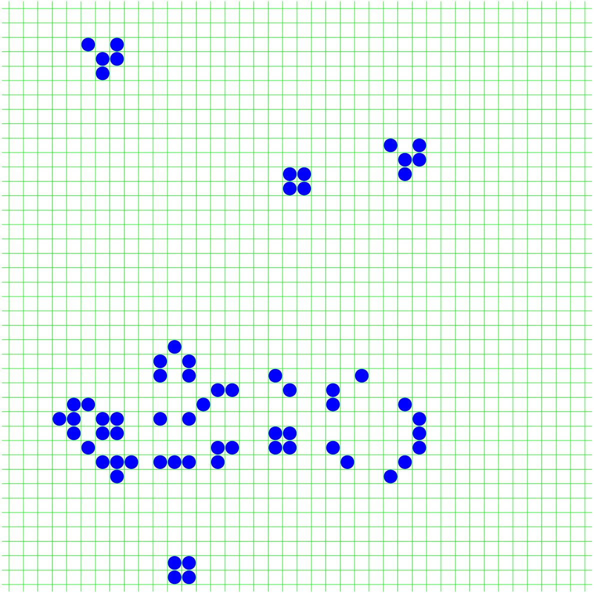

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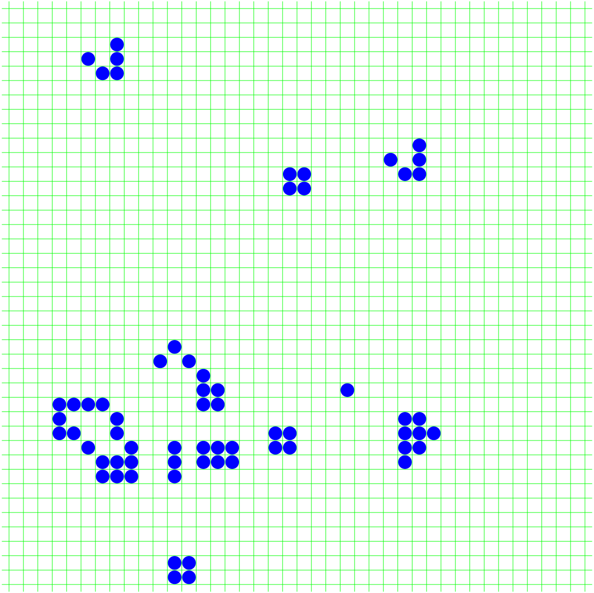

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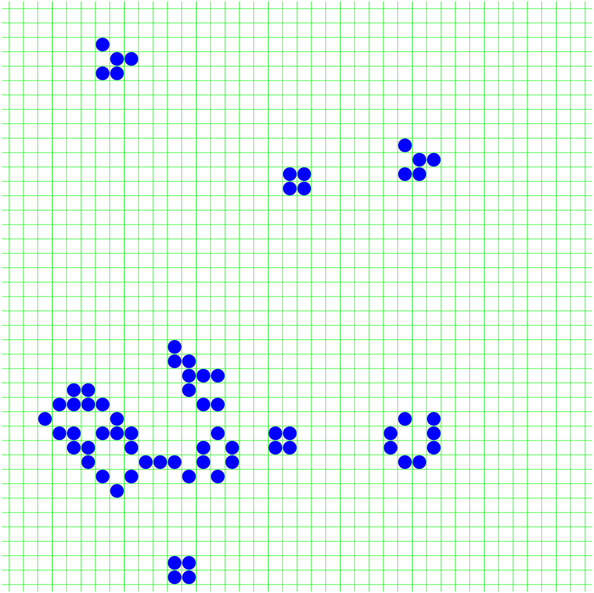

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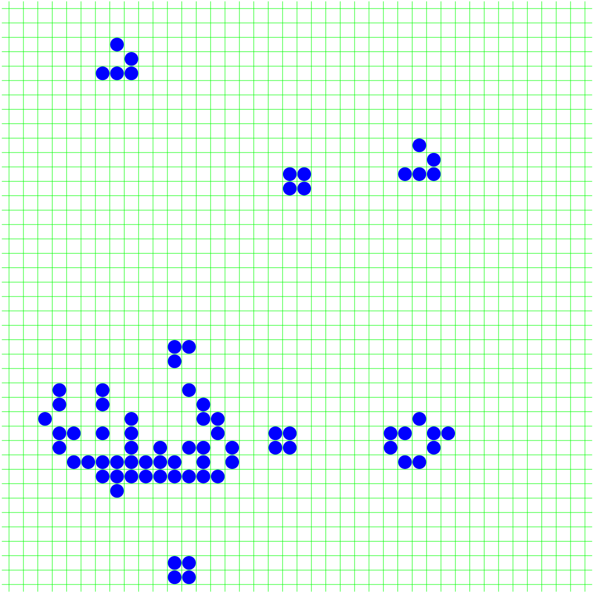

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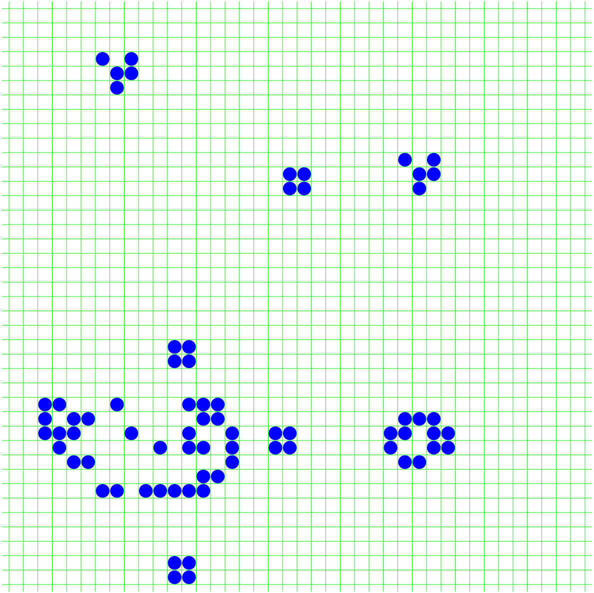

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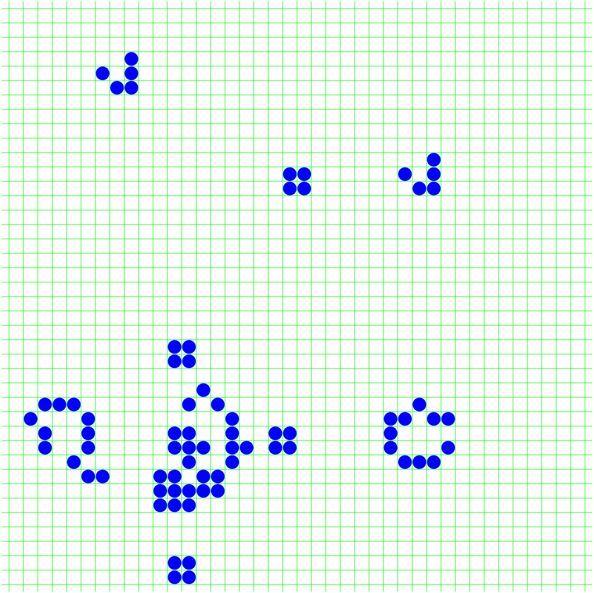

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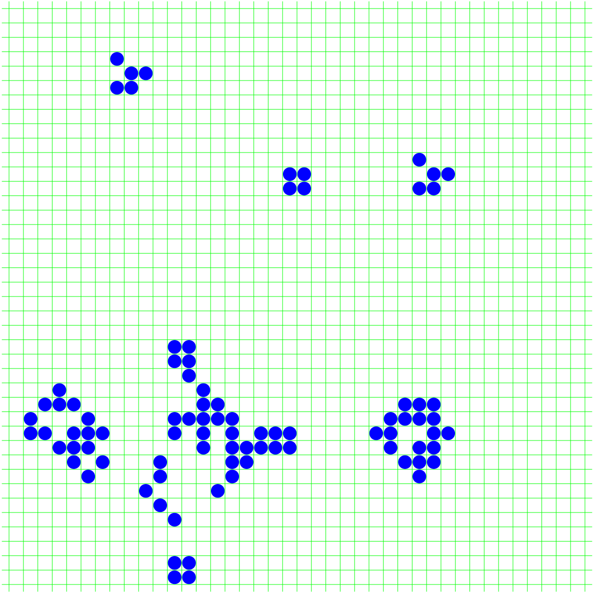

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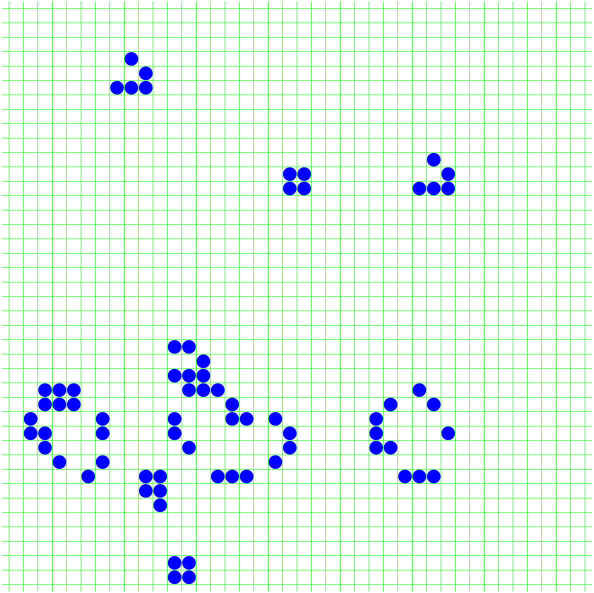

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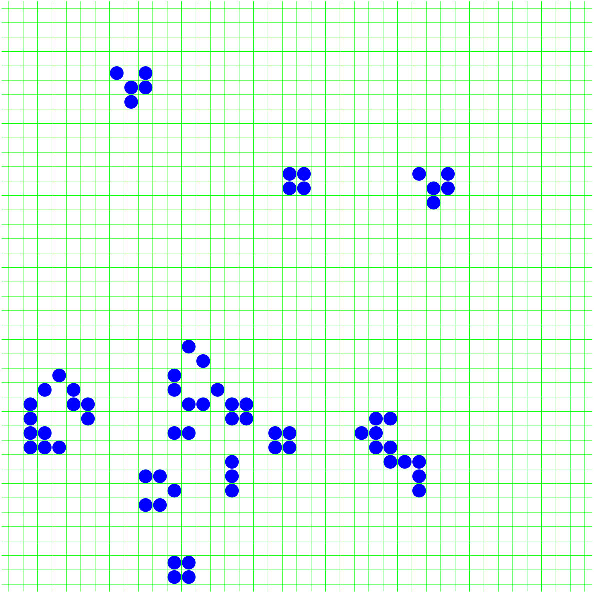

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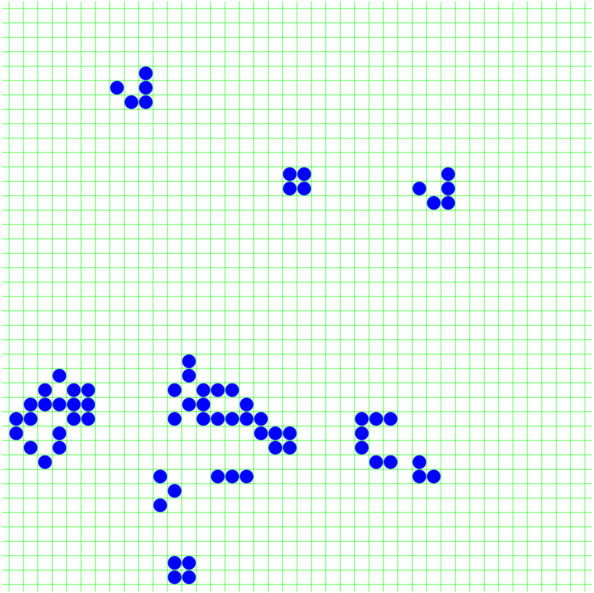

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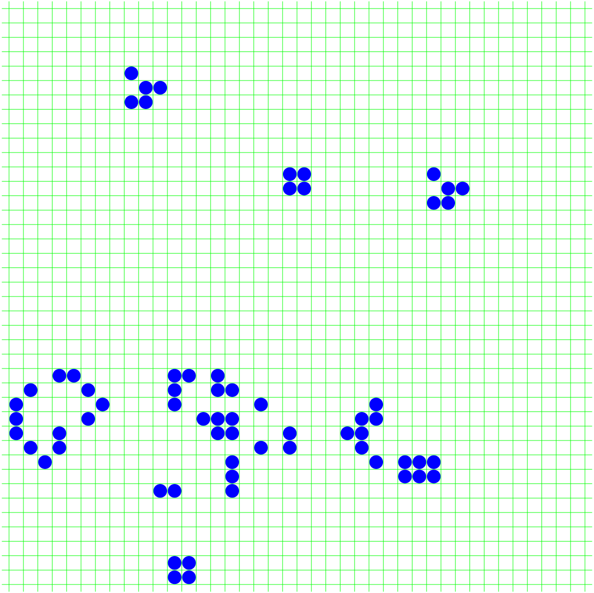

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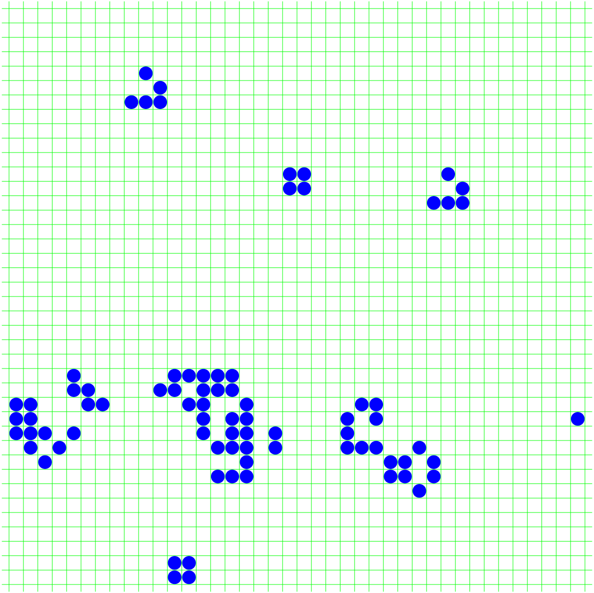

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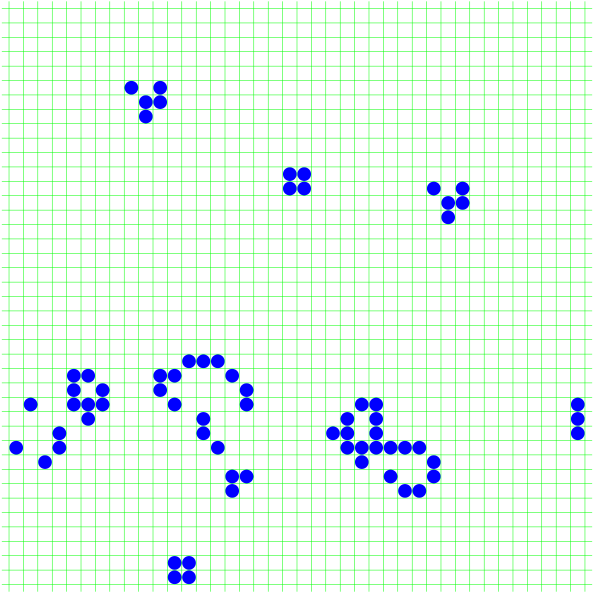

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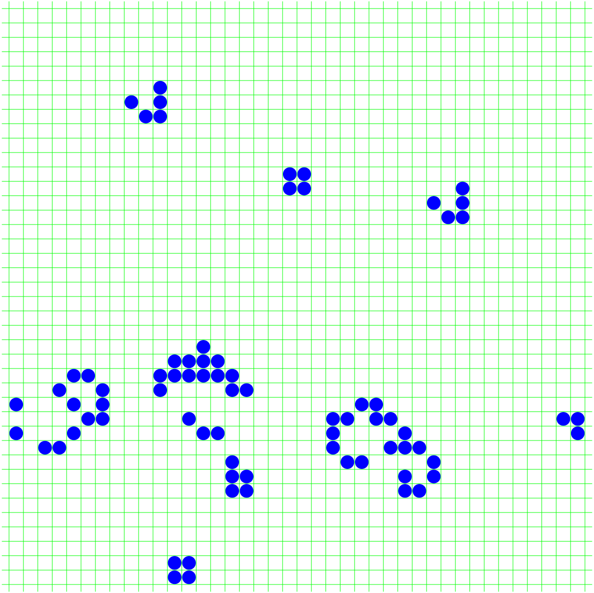

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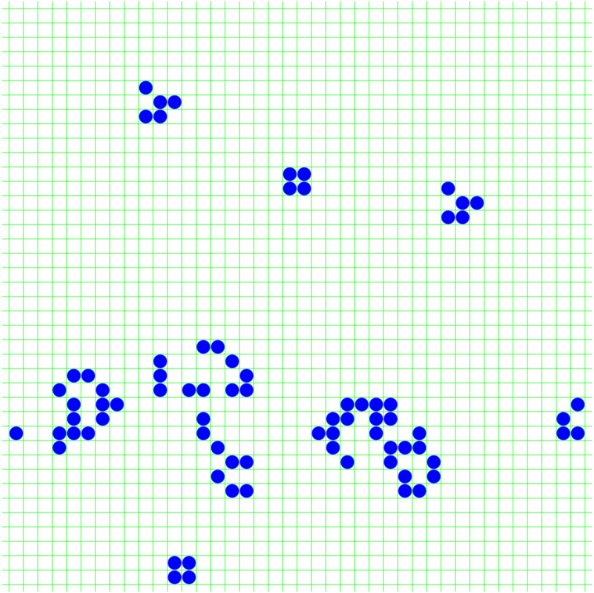

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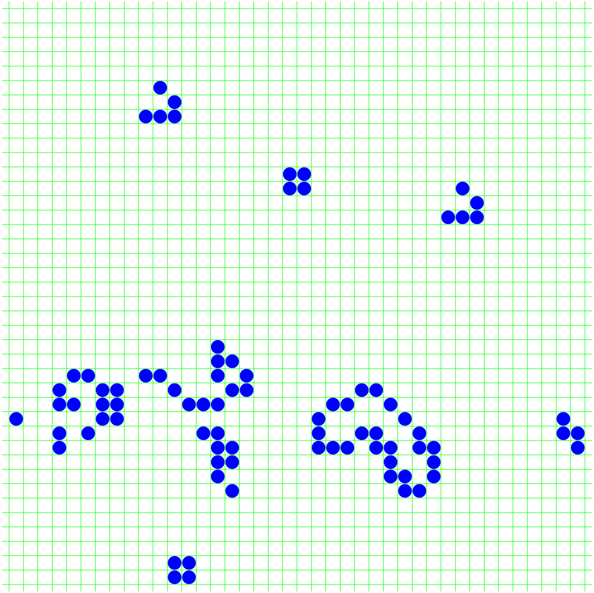

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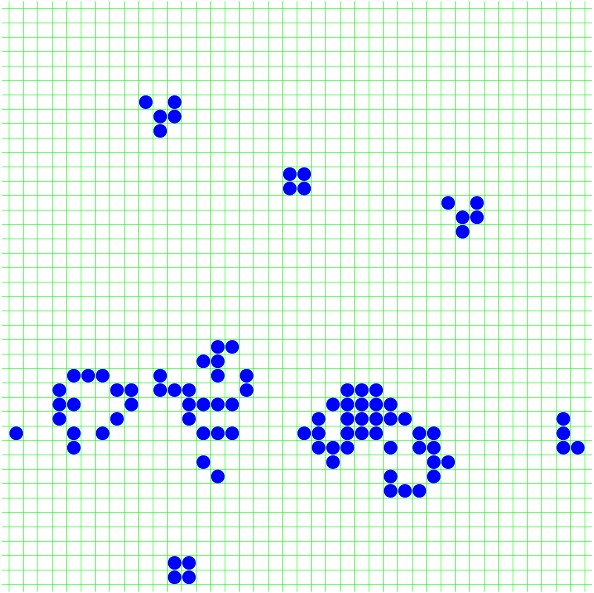

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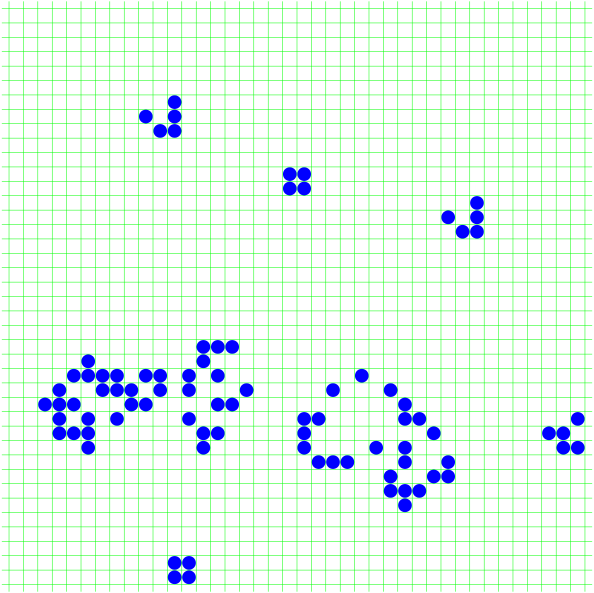

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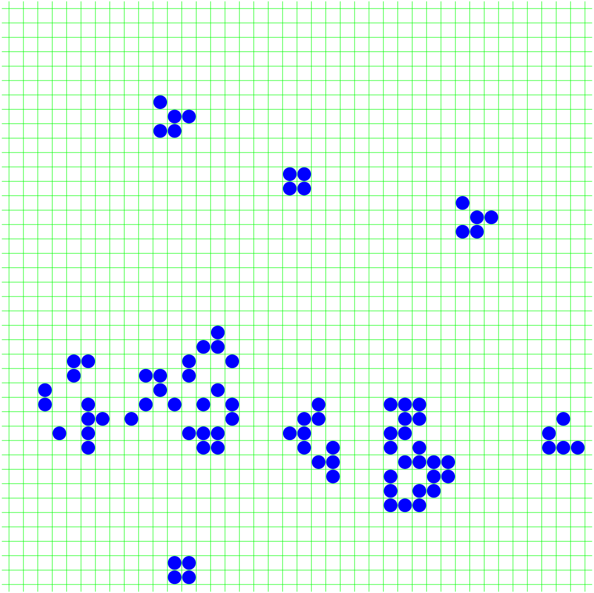

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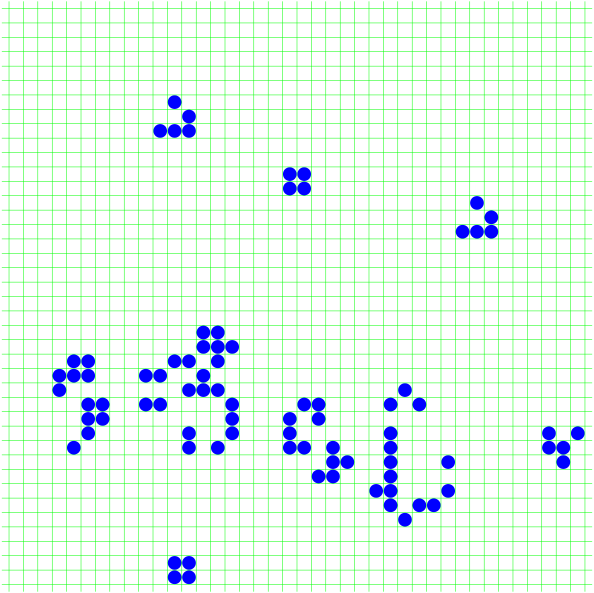

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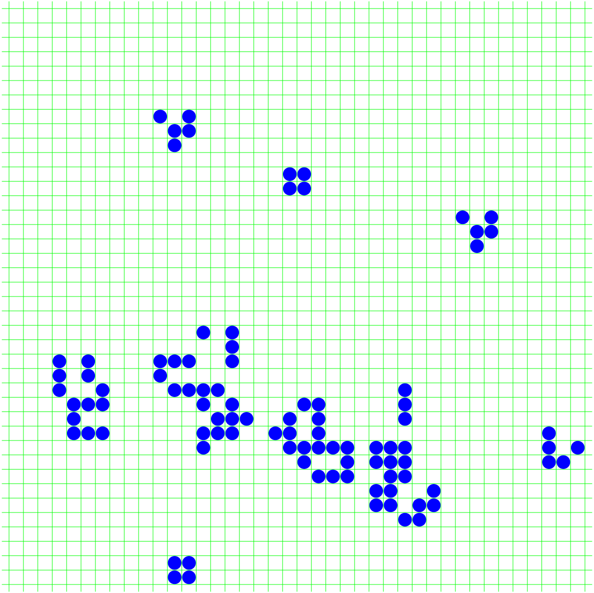

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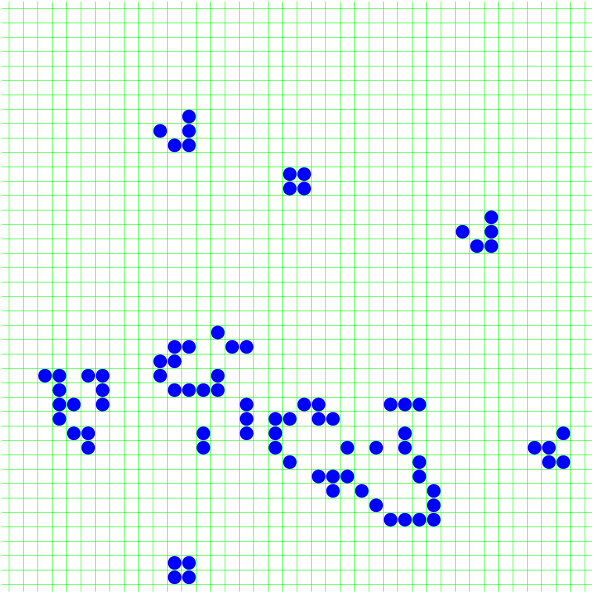

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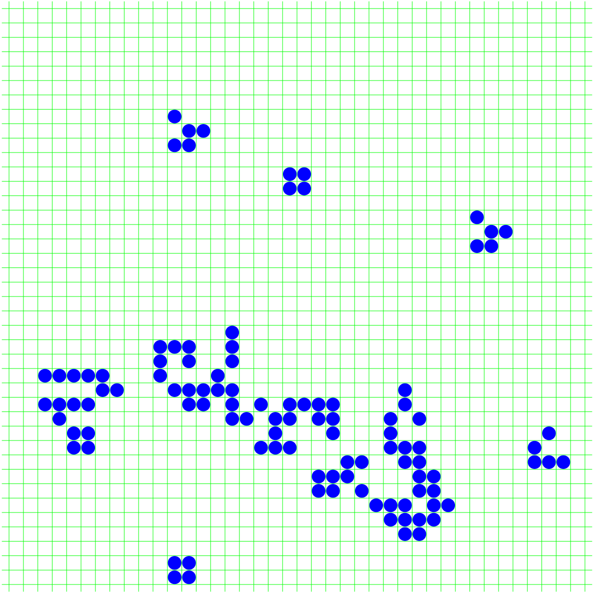

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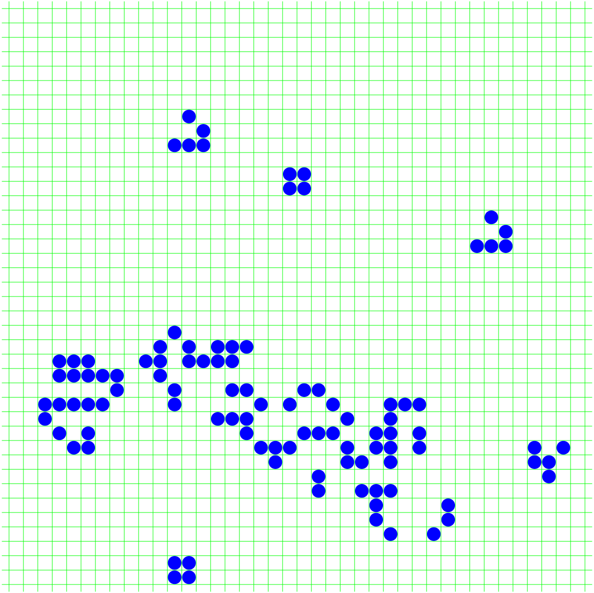
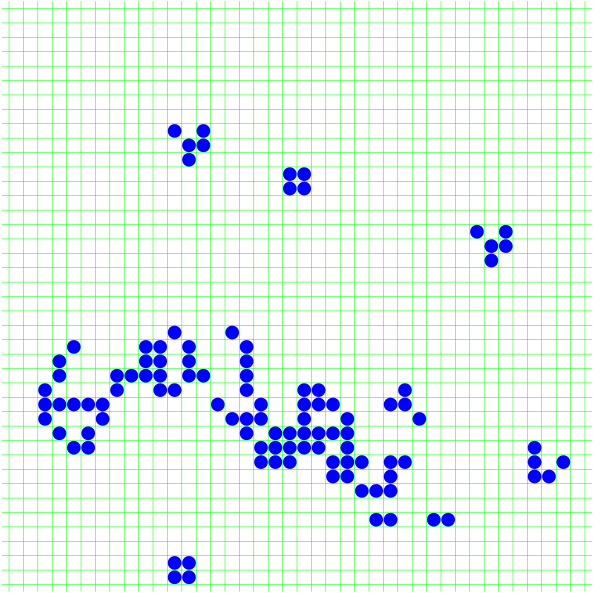
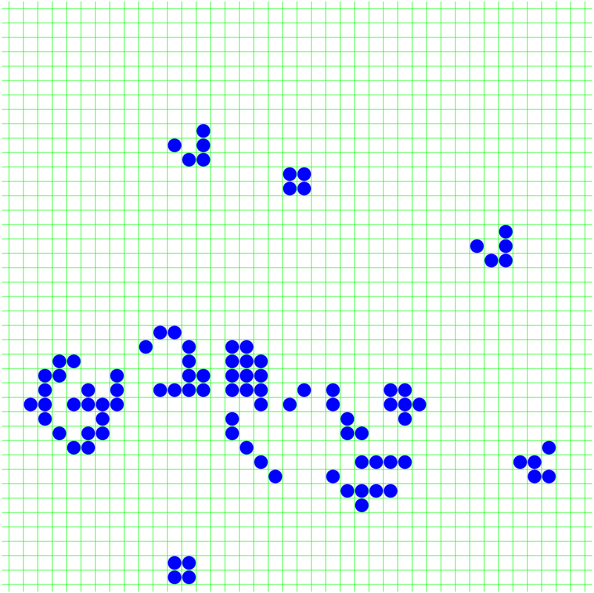


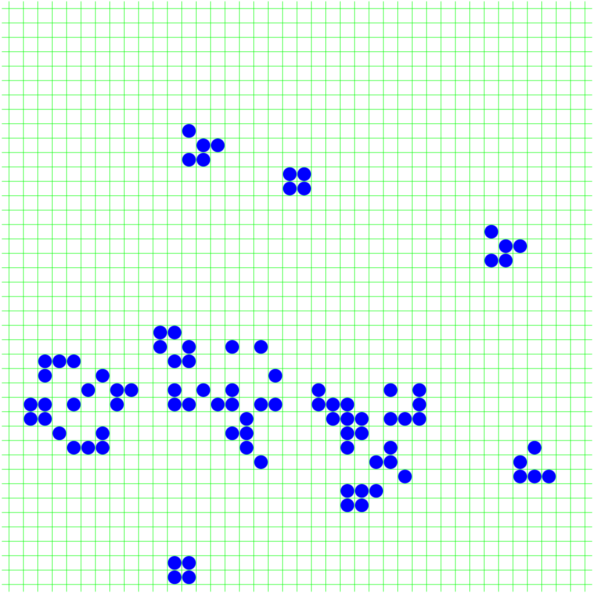
Conway's Life. Typical Evolution Producing Gliders



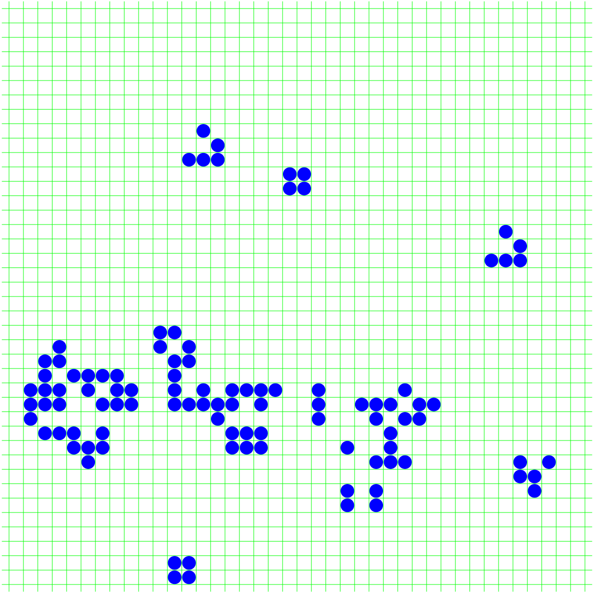
Conway's Life. Typical Evolution Producing Gliders



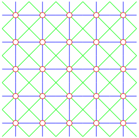
Conway's Life. Typical Evolution Producing Gliders



Conway's Life. Typical Evolution Producing Gliders



Symmetries of Toric $N \times N$ Square Lattices $\Gamma_{N\times N}$

$\Gamma_{N\times N} =$  has the same group $G_{N\times N}$ in both 4-valent von Neumann  and 8-valent Moore  cases

Principal case $N = 3, 5, 6, \ldots, \infty$ $\quad |G_{N\times N}| = 8N^2$ $\quad \mathbf{T}^2$ is normal

$G_{N\times N} = \mathbf{T}^2 \rtimes \mathbf{D}_4$ — semidirect product of $\mathbf{T}^2$ and $\mathbf{D}_4$

$\mathbf{T}^2 = \mathbb{Z}_N \times \mathbb{Z}_N$ — translation group $(\mathbb{Z}_\infty \equiv \mathbb{Z})$

$\mathbf{D}_4 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$ — dihedral group ($\mathbb{Z}_4$ – 90^0 rotations, $\mathbb{Z}_2$ – reflections)

Resembles Euclidean group $\mathbf{E}_2 = \left(\mathbf{T}^2 = \mathbb{R} \times \mathbb{R}\right) \rtimes \mathbf{O}(2)$

Exception: $N = 4$ $\quad |G_{N\times N}| = 3 \times 8N^2$ $\quad \mathbf{T}^2$ is not normal

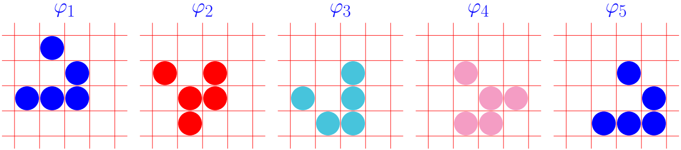
$$G_{4\times 4} = \overbrace{\left(\left(\left(\left(\mathbb{Z}_2 \times \mathbf{D}_4\right) \rtimes \mathbb{Z}_2\right) \rtimes \mathbb{Z}_3\right) \rtimes \mathbb{Z}_2\right)}^{\text{normal closure of } \mathbf{T}^2} \rtimes \mathbb{Z}_2$$

(by GAP algorithm)

Group Interpretation of “Glider”

Evolution operator = group action: $\varphi_{t_b}(x) = A_{t_a t_b}(\varphi_{t_a}(x)) = \varphi_{t_a}\left(g^{-1}_{t_a t_b} x\right)$

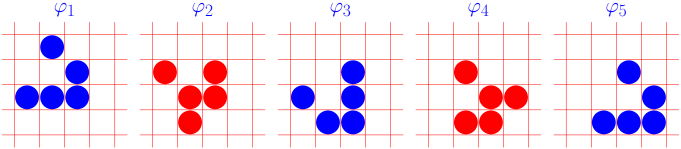
One step diagonal shift $\quad g_{15} \in \mathbf{T}^2$



“Glider” over translation group $\mathbf{T}^2$ is cycle in 4 orbits

One step downward shift → 90^o clockwise rotation → reflection in respect to vertical

$$g_{13} = g_{24} \in \mathbf{T}^2 \rtimes \mathbf{D}_4$$



“Glider” over maximal symmetry group $\mathbf{T}^2 \rtimes \mathbf{D}_4$ is cycle in 2 orbits

Two Examples of Trivalent Cellular Automata with Symmetric Local Rules

- Rule 86 $= \overbrace{01101010}^{\text{Bit string}} \sim \overbrace{\text{B123/S0}}^{\text{"Birth/Survival" lists}} \sim \overbrace{x_4' = x_4 + \sigma_3 + \sigma_2 + \sigma_1}^{\text{Polynomial over } \mathbb{F}_2}$

- Rule 23 $= 11101000 \sim$ B012/S0 $\sim x_4' = x_4\left(\sigma_2 + \sigma_1\right) + \sigma_3 + 1$

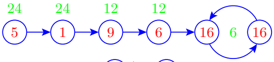
  Rule 23 is BW symmetric

Notations: $\sigma_1 = x_1 + x_2 + x_3, \;\; \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \;\; \sigma_3 = x_1x_2x_3$

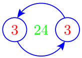
Rule 86 $\sim$ B123/S0 $\sim$ $x'_4 = x_4 + \sigma_3 + \sigma_2 + \sigma_1$

Phase Portrait Modulo Aut (“*Hexahedron*”)

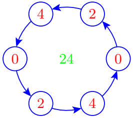
Weight of structure $p = \dfrac{\text{basin size}}{\text{number of states}}$



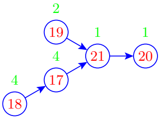
$p = \frac{39}{128} \approx 0.30$ Limit cycles



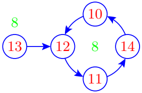
$p = \frac{3}{32} \approx 0.09$ Isolated cycles



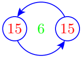
$p = \frac{9}{32} \approx 0.28$ Isolated cycles



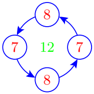
$p = \frac{3}{64} \approx 0.05$ Sink



$p = \frac{5}{32} \approx 0.16$ Limit cycles



$p = \frac{3}{128} \approx 0.02$ Isolated cycles



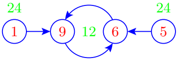
$p = \frac{3}{32} \approx 0.09$ Isolated cycles

Most cycles are spaceships — 36 of 45 or 80%

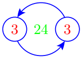
BW Symmetric Rule 23 $\sim$ B012/S0

$$\sim x_4' = x_4\left(\sigma_2 + \sigma_1\right) + \sigma_3 + 1$$

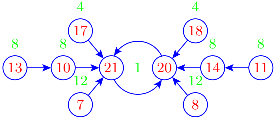
Phase Portrait Modulo Aut ("*Hexahedron*")



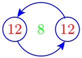
$p = \frac{9}{32} \approx 0.28$ Limit cycles



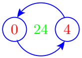
$p = \frac{3}{32} \approx 0.09$ Isolated cycles



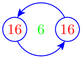
$p = \frac{33}{128} \approx 0.26$ Limit cycles



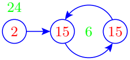
$p = \frac{1}{32} \approx 0.03$ Isolated cycles



$p = \frac{3}{16} \approx 0.19$ Isolated cycles



$p = \frac{3}{128} \approx 0.02$ Isolated cycles



$p = \frac{15}{128} \approx 0.12$ Limit cycles

$p = \frac{3}{128} \approx 0.02$ Gardens of Eden

22 spaceships of 61 cycles or about 36%

Quantum Gravity Difficulties:

- Irreversibility of Gravity: information loss (=dissipation) at horizon of black hole
- Reversibility and Unitarity of standard Quantum Mechanics

G. 't Hooft's approach to reconcile Gravity with Quantum Mechanics:

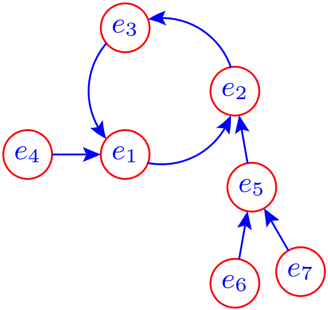
1. Discrete degrees of freedom at Planck distance scales
2. States of these degrees of freedom form primordial basis of Hilbert space with (nonunitary) evolution
3. Equivalence classes of states: two states are equivalent if they evolve into the same state after some lapse of time
4. Equivalence classes form basis of Hilbert space with unitary evolution governed by time-reversible Schrödinger equation

In our terminology this corresponds to transition to limit cycles: after short time of evolution limit cycles become physically indistinguishable from reversible isolated cycles

Primordial and Unitary Hilbert Spaces

Primordial basis

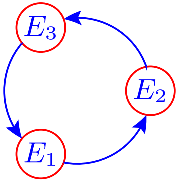
$e_1, e_2, e_3, e_4, e_5, e_6, e_7$



Equivalence classes

$E_1 = \{e_1, e_5, e_6, e_7\}$
$E_2 = \{e_2\}$
$E_3 = \{e_3, e_4\}$

form unitary basis



Why irreversibility is not visible: time out of cycle $\approx 10^{-44}$ sec (Planck time), time on cycle potentially ∞, minimal time fixed in today's experiments $\approx 10^{-18}$ sec $\approx 10^{26}$ Planck units

Search For Reversibility

Reversibility is rare property tending to disappear with growth of complexity of lattice

Two rules trivially reversible on all lattices

- 85 $\sim$ B0123/S $\sim$ $x_4' = x_4 + 1$
- 170 $\sim$ B/S0123 $\sim$ $x_4' = x_4$

Tetrahedron, 6 additional reversible rules

- 43 $\sim$ B0/S012 $\sim$ $x_4' = x_4\left(\sigma_2 + \sigma_1\right) + \sigma_3 + \sigma_2 + \sigma_1 + 1$
- 51 $\sim$ B02/S02 $\sim$ $x_4' = \sigma_1 + 1$
- 77 $\sim$ B013/S1 $\sim$ $x_4' = x_4\left(\sigma_2 + \sigma_1 + 1\right) + \sigma_3 + \sigma_2 + 1$
- 178 $\sim$ B2/S023 $\sim$ $x_4' = x_4\left(\sigma_2 + \sigma_1 + 1\right) + \sigma_3 + \sigma_2$
- 204 $\sim$ B13/S13 $\sim$ $x_4' = \sigma_1$
- 212 $\sim$ B123/S3 $\sim$ $x_4' = x_4\left(\sigma_2 + \sigma_1\right) + \sigma_3 + \sigma_2 + \sigma_1$

Cube, 2 additional reversible rules

- 51 $\sim$ B02/S02 $\sim$ $x_4' = \sigma_1 + 1$
- 204 $\sim$ B13/S13 $\sim$ $x_4' = \sigma_1$

Dodecahedron and Graphene 6$\times$4, no additional reversible rules

Mesoscopic Lattice Models

- Non-deterministic dynamical system — transition from a state to any other state is possible with some probability
- Markov chain is typical example
- Lattice models are special instances of Markov chains
- Stationary distributions of these Markov chains is subject of statistical mechanics
- Mesoscopic systems — too large for detailed microscopic description but too small for classical thermodynamics
- Mesoscopy studies nuclei, atomic clusters, nanotechnological structures, multi-star systems etc.

Canonical and Microcanonical Ensembles

Gibbs canonical ensemble with canonical partition function

$$Z = \sum_{\sigma\in\Sigma} \mathrm{e}^{-E_\sigma/k_B T}$$

— main tool of conventional statistical mechanics — being essentially asymptotic concept based on “thermodynamic limit” approximation can not be applied to mesoscopic systems

Instead more fundamental microcanonical ensemble

with entropy formula

$$S_E = k_B \ln \Omega_E$$

or, equivalently, with microcanonical partition function

$$\Omega_E = \mathrm{e}^{S_E/k_B}$$

Symmetry Approach To Mesoscopic Lattice Models
is based on exact enumeration of group orbits of microstates

Advantages

- Statistical studies are based on simplifying assumptions, it is important to control these assumptions by exact computation, wherever possible.
- Exact computation might reveal subtle details in behavior of systems.

Example: Ising model

- Vertex values: $s_i \in Q = \{-1, 1\}$
- Hamiltonian: $H = -J \sum_{(i,j)} s_i s_j - B \sum_i s_i$

Microcanonical Partition Function

Ising Model On Dodecahedron

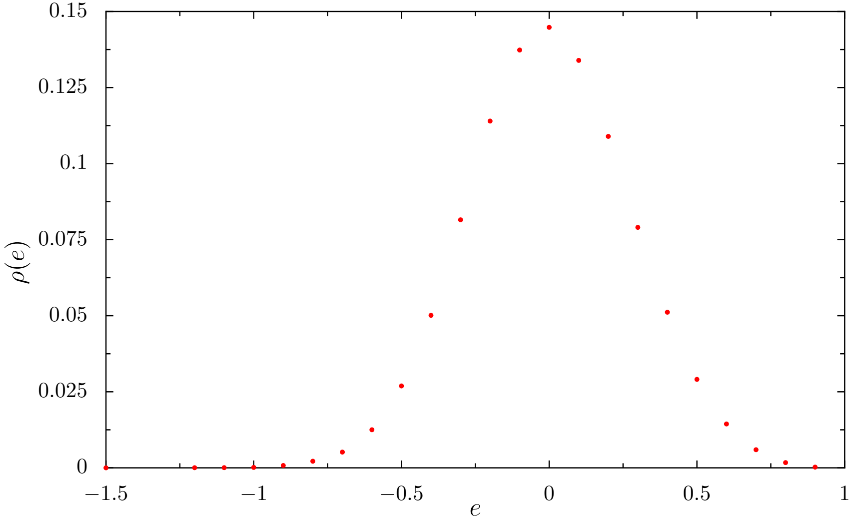


Figure: Density of states $\rho(e) = \Omega_E/\Omega$ vs. energy per vertex $e = E/\left|V(\Gamma)\right|$

Magnetization

Ising Model On Dodecahedron

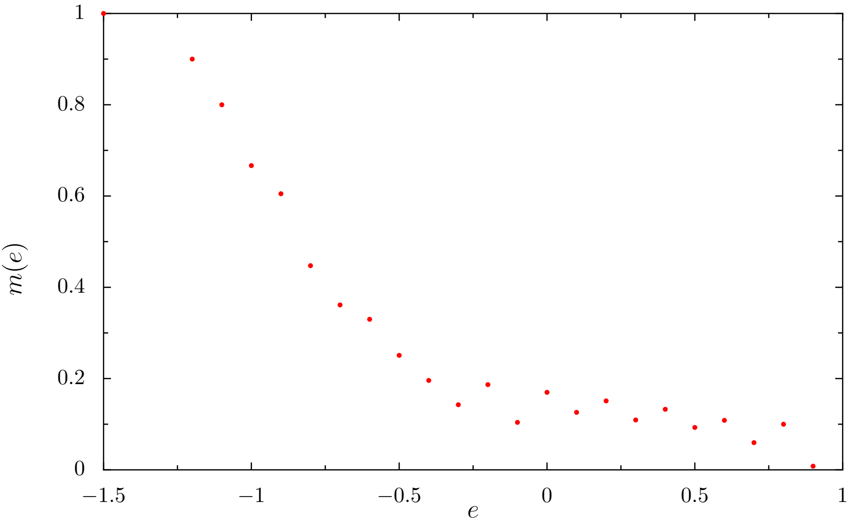


Figure: Specific magnetization $m(e) = M(E)/|V(\Gamma)|$ vs. energy per vertex e

Phase Transitions In Mesoscopic Systems
"Convex Intruders"

In standard thermodynamics

$\left.\frac{\partial^2 S}{\partial E^2}\right|_V = -\frac{1}{T^2}\frac{1}{C_V} \Longrightarrow$ entropy vs. energy diagramm always concave
C_V is specific heat

In microcanonical thermodynamics

(mesoscopic and nonextensive systems) there are energy intervals called "convex intruders" where $\left.\frac{\partial^2 S}{\partial E^2}\right|_V > 0$
Convex intruders are indicators of first-order phase transitions

Convex intruders in discrete case can be found

via easily computed inequality $\Omega^{p+q}_{E_i} < \Omega^{p}_{E_{i-1}}\Omega^{q}_{E_{i+1}}$
where $p/q = \left(E_{i+1} - E_i\right)/\left(E_i - E_{i-1}\right)$

Convex Intruders In Ising Model On Dodecahedron

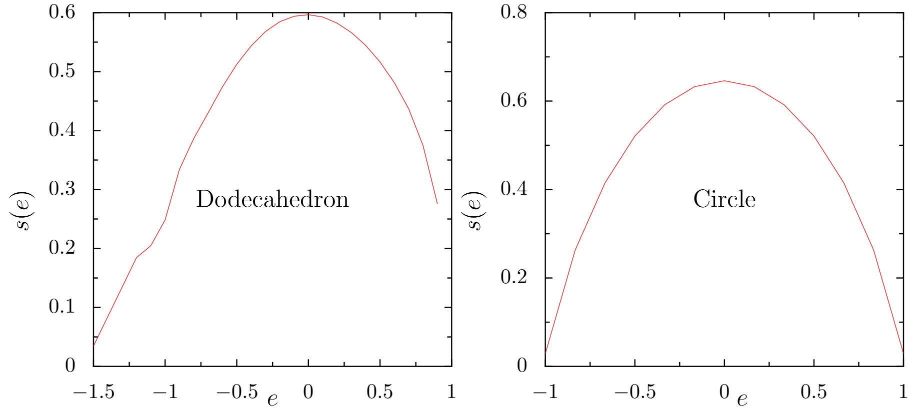


Figure: Microcanonical entropy $s(e) = \ln(\Omega_E) / |V(\Gamma)|$ vs. energy e. Left diagram contains distinct convex intruder in interval $-1.2 \leq e \leq -0.9$ and subtle one in interval $-0.8 \leq e \leq -0.6$. Right diagram is fully concave: 1D Ising model has no phase transitions.

Convex Intruders. 3-, 4- and 6-valent Lattices

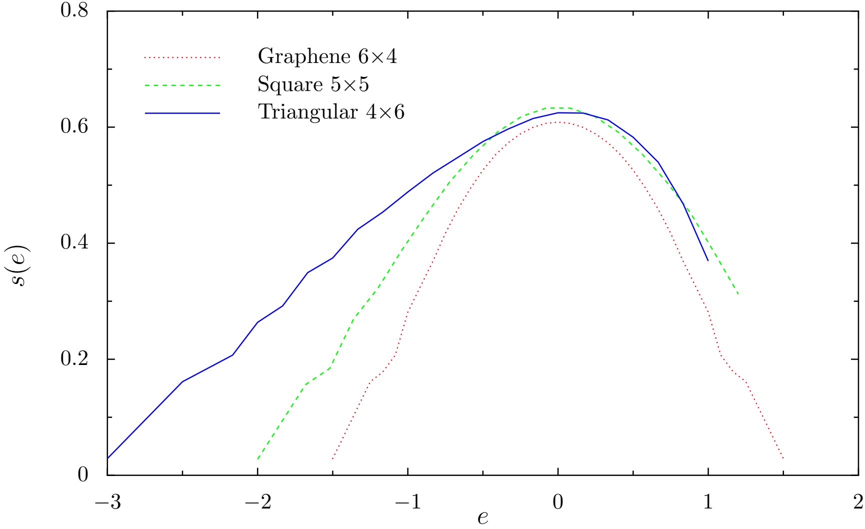


Figure: Entropy on 3-valent (dot), 4-valent (dash) and 6-valent (solid) tori

Main Results

- C program for symmetry analysis of finite discrete dynamical systems has been created
- Trajectories of dynamical systems with non-trivial symmetry go always in the direction of non-increasing sizes of group orbits. Cyclic traectories lie within orbits of the same size
- Evolution operators can be replaced by group shifts after some lapse of time. Initial data tend to form soliton-like structures
- Reversibility is rare property. Reversible systems are trivial
- Lattice symmetries facilitate study (in particular, search of phase transitions) of physical lattice models