Robust Stability for Parametric Linear ODEs

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Content: A logical tool in robust control theory for systems of parametric inhomogeneous linear ODEs.

Topic: systems of linear ODEs with parametric constant coefficients and parametric exponential polynomials as inhomogeneities.

Goal: Find necessary and sufficient conditions on number and function parameters in systems of linear ODEs that guarantee certain stability conditions and/or initial value conditions on solution functions.

Function domain: exponential polynomials

Logical Framework:

We study of boolean combinations of implicit multivariate linear ODEs of arbitrary order with parametric complex constant coefficients and parametric inhomogeneuos parts with additional functions and predicates referring to properties of function-variables together with arbitrary complex polynomial equations and inequalities (with order relation restricted to real arguments).

Function Domain:

D consists of **complex exponential polynomials,** i. e. complex polynomials in the independent realvariable x and in $\exp(\lambda x)$ for arbitrary complex values of λ.

They have a **unique representation** in the form

$$f := \sum_{\alpha \in S} p_\alpha(x) \exp(\alpha x)$$

with non-zero complex polynomials $p_\alpha(x)$ and S a finite set of complex numbers.

We call S the **spectrum** $\text{spec}(f)$ of f, and $|S|$ the **specsize** $specsize(f)$ and the maximal degree of all $p_\alpha(x)$ the **degree** $deg(f)$ of f. We refer to the elements α of $\text{spec}(f)$ and the coefficients of the correponding polynomials $p_\alpha(x)$ as the **numerical data** of f. We call a function $f \in D$ **constrained** by some $b \in \mathbb{N}$ if b is a common upper bound for all numbers $specsize(f)$ and $deg(f)$. If this is the case, then the number of numerical data for f is bounded by $(b+1)b$.

The **critical spectrum** of f is the set of all $\alpha \in spec(f)$ with $\Re(\alpha) = 0$, and the **safe spectrum** of f is the set of $\alpha \in spec(f)$ with $\Re(\alpha) < 0$. Astab(f) holds if f is **asymptotically stable** i.e. if the whole spectrum of f is safe. Stab(f) holds if f is **stable** i.e. if the whole spectrum of f is safe or critical, and for all critical α the corresponding polynomial p_α is constant.

The classical results of **Routh-Hurwitz** and **Lienard-Chipart** give quantifier-free conditions in terms of signs of determinants in the (constant) coefficients of a system of homogeneous ODEs for all solutions to be stable or asymptotically stable.

First Main Result:

We show that the solvability of such a system of conditions including initial value conditions and/or positive and negative stability conditions can be equivalently reduced to a boolean combination of polynomial equations and inequalities in the parametric number-coefficients, and the following numerical data of the function parameters u:
The elements α of $spec(u)$ and the coefficients of the corresponding polynomials $p_\alpha(u)$.
Proviso: All function-parameters u have to be constrained.

The reduction is achieved by an **constrained algorithmic quantifier elimination (QE) procedure** using non-linear real and complex QE-algorithms.

The **non-linearity** is in contrast to the situation without stability conditions that can be handled by linear differential elimination methods within domains of germs of meromorphic functions (W. CASC 2005).
The result can be construed as a strong generalization of the classical criteria of **Routh-Hurwitz** and **Lienard-Chipart**.

Contrasting Warning: We show that even seemingly simply problems on ODEs with parametric non-constant coefficients turn out to be algorithmically undecidable in the domain of holomorphic functions.

Second Main Result:

Sample solutions of such a parametric system of conditions can be represented symbolically uniformly in the parametric number-coefficients, and the numerical data of the function parameters u.

Technical Details:

The **logical framework** is as follows:

We have **two sorts** of variables, the F-variables ranging over D and the N-variables ranging over $\mathbb{C}$. In the N-sort we have constants for all rational numbers and for $I := \sqrt{-1}$, the ring operations $+, -, \cdot$, the operations $\Re$ and $\Im$ and the order relation (restricted to $\mathbb{R}$).

In the F-sort we have the operations $+, -, '$ and for every natural number b the unary predicates $Specsize_b(y)$ and $Deg_b(y)$.

In addition we have mixed function symbols for scalar multiplication of an N-term with an F-term, and for all natural numbers i, j unary function-symbols $spec_i(y)$ and $spec_{i,j}(y)$ mapping $F-$terms into N-terms.

Semantics: $spec_i(y)$ denotes the i-th element of the spectrum of y in the lexicographical order. $spec_{i,j}(y)$ denotes the j-th coefficient of the polynomial $p_\alpha(x)$ belonging to $\alpha = spec_i(y)$ in the unique representaion of y. In the exceptional case $i > specsize(y)$ we put both values to zero.

Atomic formulas are equations $s = t$ between two F-terms s, t, or equations $s = t$ or inequalities $s < t$ between two N-terms s, t, or finally predicates $Specsize_s(t)$ and $Deg_d(t)$ for an F-term t.

The first type represents **parametric inhomogeneous implicit linear ODEs,** the second type **complex polynomial equations of inequalities** and the third type **constraints of function parameters.**

Quantifier-free formulas are arbitrary combinations of atomic formulas by $\wedge$ (and), $\vee$ (or), $\neg$ (not).

In the formation of **arbitrary formulas** we allow in addition arbitrary quantification $\exists y$, $\forall y$ over F-variables and over N-variables $\exists \xi$, $\forall \xi$ - **provided** ξ does not occur in any atomic formula of the F-sort. **Constrained formulas** allow only quantification over explicitly constrained function variables.

Expressive power of quantifier-free formulas:
They can express:

- parametric initial value conditions on functions and their derivatives
- global conditions on stability or asymptotic stability
- local stability conditions on functions.

A **Constrained QE-procedure** computes for every constrained input formula an equivalent constrained quantifier-free formula under the hypothesis that all function parameters are explicitly constrained.

Main technical result: **Constrained QE**

Theorem 1 *There is a constrained QE-procedure for $(\mathbb{C}, D)$ in this language. Specifically: For every natural number b and every constrained formula $\varphi(\eta_1, \ldots, \eta_m, u_1, \ldots u_n)$ one can compute a quantifier-free formula $\varphi'_b(\eta_1, \ldots, \eta_m, u_1, \ldots u_n)$ such that in $(\mathbb{C}, D)$ the following holds:*

$$\bigwedge_{i=1}^{n}(Specsize_b(u_i) \ \wedge \ Deg_b(u_i)) \ \longrightarrow$$

$$(\varphi(\eta_1, \ldots, \eta_m, u_1, \ldots u_n) \ \longleftrightarrow \varphi'_b(\eta_1, \ldots, \eta_m, u_1, \ldots u_n))$$

Moreover, if the input formula is purely existential, then it may also contain unconstrained quantifiers.

Second technical result: **Extended Constrained QE**

Theorem 2 *For an existential input formula with constrained function parameters one can construct a finite system of pairs of quantifier-free formulas and formal expressions in the parameters that serve as sample solutions for the quantified variables. These formulas form a complete case distinction and the corresponding formal expressions are the numerical data of a solution in the given case.*

Example: Harmonic Oscillator

Let $b = 1$ and let $\varphi(\eta, u)$ for a complex number-variable η and a function-variable u be the following formula φ :

$$\exists y(my'' + cy' + ky = u \ \wedge \ y \neq 0 \wedge \ Stab_2(y))$$

Then $Specsize_1(u) \ \wedge \ Deg_1(u)$ implies the equivalence of φ with explicitly computable quantifier-free formula φ' in the entities

$$m, c, k, spec_1(u), spec_{1,0}(u), spec_{1,1}(u)$$

that is necessary and sufficient for the stability of a solution.
In particular it analyses, when resonance occurs.
Moreover - modulo a finite case distinction - we get solutions given by exponential polynomials with numerical data given by formal expressions in the same parameters

$$m, c, k, spec_1(u), spec_{1,0}(u), spec_{1,1}(u).$$

Main steps of the algorithm:

Elimination of a number-quantifier: Use a real QE-procedure extended to complex numbers.

Elimination of a function-quantifier $\exists y$ in front of a quantifier-free formula with constrained free function variables.

Reduce the given formula to a conjunction.

Reduce to at most one positive occurrence of an ODE containing y by order reduction with case distinctions.

Replace $\exists y$ by a tuple of existential number quantifiers describing the numerical data of y i.e. the possible spectrum and coefficients of associated polynomials in the canonical representation of a possible solution. Here one has to choose an appropriate constraining number b for y depending on the constraints for the function-parameters and on the order of the positive ODE in y. Be careful to choose b large enough to cover the resonance case!.

This reduces ODEs containing y to a conjuction of non-linear polynomial equations in the new number variables and expressions of the form $spec_i(t)$, $spec_{i,j}(t)$ for F-terms containing function parameters. At this point the elimination of number-quantifiers is called in order to eliminate the quantifiers w.r.t. the newly introduced number variables.

An extended real and complex QE for number quantifiers yields now a corresponding extended constrained QE for function quantifiers, where functions are represented by their parametric numerical data.

What about **parametric ODEs with non-constant coefficients**?
Works well in domains of germs of meromorphic functions without stability conditions (W. CASC 2005). The present note answers a question of **A. Weber** at CASC 2005.

But **Warning** for the domain of holomorphic functions:

Theorem 3 *Let K be a subfield of $\mathbb{C}$ and let R be a differential subring of $K[[X]]$ in the language $L := \{0, 1, X, +, -, \cdot, '\}$. Consider systems φ of linear homogeneous ODEs and of polynomial equations over K. Then the solvability of φ in (R, K) is an undecidable problem.*

Proof. The system of linear differential equations $X \cdot y' = a \cdot y,\ y \neq 0,\ a' = 0$ has a solution y in R iff $a \in \mathbb{N}$. So one can code Hilbert's 10. problem. $\Box$

Open Problems:

- What happens with decidability in this theorem, if one drops the constant "X'' and/or restricts the polynomial equations to linear ones?
- What is the asymptotic complexity of the constrained QE procedure?
- Is the constrained QE procedure practically feasable (e.g. in REDLOG)?