Randomized Rounding

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Overview

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- Using only randomness is not enough: (BPP = P)?
- Error bounds in deterministic approximation are often bad or hard to estimate.

Goals

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- Fast.
- With high probability: Close to an optimal solution.

Performance Ratio of a Randomized Algorithm

Let x be an instance of the Problem P. Let A be an algorithm for solving P, let $m_A(x)$ denote the size of the solution produced by A on x and let $m^*(x)$ denote the size of an optimal solution. The performance ratio is given by

$$\max\{\sup_{x\in P} \frac{m^*(x)}{m_A(x)}, \inf_{x\in P} \frac{m_A(x)}{m^*(x)}\}. \tag{1}$$

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If A uses randomization: $\mathbb{E}[m_A(x)]$

Overview

Boole's inequality

Theorem (Boole's inequality)

Let (A_i) be any countable set of events. Then

$$\mathbb{P}\left[\bigcup_i A_i\right] \leq \sum_i \mathbb{P}\left[A_i\right]. \tag{2}$$

Chernoff-Bound variant

Theorem (Chernoff-Bound variant)

Let $X_1, \ldots, X_n$ be a sequence of independent Bernoulli trials, such that $\mathbb{P}[X_i = 1] = p_i$ and $\mathbb{P}[X_i = 0] = 1 - p_i$. Let S be a subset of $\{1, \ldots, n\}$ and let $s = |S|$. Define $X = \sum_{i \in S} X_i$. Then

$$\mathbb{P}\left[|X - \mathbb{E}[X]| > \sqrt{4s \ln s}\right] \leq \frac{1}{s^2}. \qquad (3)$$

□

Linear Programming

A Linear Program is a problem of the following type:

$$
\begin{array}{rr}
\text{Maximize} & \mathbf{c}^T\mathbf{x} \\
\text{constrained by} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
\text{for} & \mathbf{c}, \mathbf{b} \in \mathbb{Q}^n, \mathbf{A} \in \mathbb{Q}^{n \times n}
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While Linear Programs are efficiently solvable, Integer Linear Programming (ILP) is known to be **NP** complete.

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2. Relax the integrality requirement and compute an optimal solution to the corresponding LP.
3. Approximate the optimal LP-solution to obtain an integer-approximation of the ILP-solution.

Randomized Rounding: Analysis

Let $P = (\mathbf{A}, \mathbf{c}, \mathbf{b})$ be an ILP, let $\mathbf{x} = (x_1, \ldots, x_n)$ denote the variable vector of P and let $\widehat{\mathbf{x}} = (\widehat{x_1}, \ldots, \widehat{x_n})$ denote a solution to the corresponding Linear Program.

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For each $i = 1, \ldots, n$ define $\overline{x_i}$ equal to 1 with probability $\widehat{x_i}$ and equal to 0 otherwise. Let $\overline{\mathbf{x}} = (\overline{x_1}, \ldots, \overline{x_n})$ be the corresponding vector of random variables.

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Then for any row vector $\mathbf{a}$ of $\mathbf{A}$

$$\mathbb{E}[\mathbf{a} \cdot \overline{\mathbf{x}}] = \mathbf{a} \cdot \widehat{\mathbf{x}}.$$

Overview

Lattice Approximation: Problem Statement

Let $\mathbf{A} \in \mathbb{B}^{n \times n} = \{0, 1\}^{n \times n}$ be a $n \times n$ matrix with entries in $\{0, 1\}$ and $\mathbf{p} \in [0, 1]^n$ a vector with entries in the real interval $[0, 1]$.

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This is already stated as LP!

Lattice Approximation: Solution step 2

$$\min_{\mathbf{q}\in\mathbb{F}_2^n} \|\mathbf{A}\cdot(\mathbf{p}-\mathbf{q})\|_\infty$$

- Step 2: Solve the non-integral version of the problem.

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- Step 2: Solve the non-integral version of the problem.
- Simply choose $\mathbf{q} = \mathbf{p}$.

Lattice Approximation: Solution step 3

$$\min_{\mathbf{q} \in \mathbb{F}_2^n} \|\mathbf{A} \cdot (\mathbf{p} - \mathbf{q})\|_\infty$$

- Choose $\overline{\mathbf{q}}$ with $\overline{q_i} = 1$ with probability p_i as solution.

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- $\mathbb{E}[\mathbf{A_i}\cdot\overline{\mathbf{q}}] = \mathbf{A_i}\cdot\mathbf{p}$
- Chernoff:

$$\mathbb{P}\left[|\mathbf{A_i}\cdot\overline{\mathbf{q}} - \mathbf{A_i}\cdot\mathbf{p}| \geq \sqrt{4n\ln n}\right] \leq \frac{1}{n^2}.$$

Lattice Approximation: Solution analysis

$$
\begin{aligned}
\mathbb{P}\left[\|\mathbf{A}\cdot(\overline{\mathbf{q}}-\mathbf{p})\|_\infty > \sqrt{4n\ln n}\right] &= \mathbb{P}\left[\bigcup_i |\mathbf{A_i}\cdot\mathbf{q}-\mathbf{A_i}\cdot\mathbf{p}| > \sqrt{4n\ln n}\right] \\
&\le \sum_i \mathbb{P}\left[|\mathbf{A_i}\cdot\overline{\mathbf{q}}-\mathbf{A_i}\cdot\mathbf{p}| > \sqrt{4n\ln n}\right] \\
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Thus with probability $1-\frac{1}{n}$ we find a solution $\overline{\mathbf{q}}$ with $\|\mathbf{A}\cdot(\overline{\mathbf{q}}-\mathbf{p})\|_\infty \leq \sqrt{4n\ln n}$.

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MaxSAT: Problem Statement

Let I be an instance of SAT, which without loss of generality is assumed to be in conjunctive normal form (CNF), i.e. a set C of clauses in CNF over a set of boolean variables V.

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Let I be an instance of SAT, which without loss of generality is assumed to be in conjunctive normal form (CNF), i.e. a set C of clauses in CNF over a set of boolean variables V.
A solution to this problem is an assignment $A : V \to \mathbb{B}$, such that the number of clauses $m_A(I)$ that evaluate to true is maximal.

MaxSAT: Solution Step 1

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$$
\begin{aligned}
\text{Maximize} \quad & \sum_j z_j \\
\text{constrained by} \quad & \sum_{y_i \in S_j^+} y_i + \sum_{y_i \in S_j^-} (1 - y_i) \geq z_j \forall j \\
\text{and constrained by} \quad & y_i, z_j \in \{0, 1\}.
\end{aligned}
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1. Randomized Rounding. Works well for clauses with few literals:

Lemma

Let C_j be a clause with k literals. Then for the probability p_j that C_j evaluates to true, the following holds:

$$p_j \geq (1 - (1 - \frac{1}{k})^k)\widehat{z}_j.$$

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2. Assign truth values with equal probability. Works well for large clauses:

$$p_j \geq (1 - \frac{1}{2^k}).$$

MaxSAT: Solution Analysis

Have:

- Randomized Rounding for small clauses.

$$p_j \geq (1 - (1 - \frac{1}{k})^k)\widehat{z}_j.$$

- Random truth assignment for large clauses.

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Simply running both algorithms and choosing the better output already leads to a $\frac{4}{3}$-approximation.

k	$(1 - \frac{1}{2^k})$	$(1 - (1 - \frac{1}{k})^k)$
1	$\frac{1}{2}$	1
2	$\frac{3}{4}$	$\frac{3}{4}$
≥ 3	$\geq \frac{7}{8}$	$\geq 1 - \frac{1}{e} \geq \frac{5}{8}$

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$$\max\{m_1, m_2\} \geq \frac{1}{2}(m_1 + m_2) \tag{4}$$

$$\geq \sum_{k\geq 1} \sum_{c_j \in C_k} \frac{\alpha_k + \beta_k}{2} \hat{z}_j \tag{5}$$

$$\geq \frac{3}{4} \cdot OPT \tag{6}$$

Overview

Semidefinite Programming(1)

$$\min_{x^1,\ldots,x^n \in \mathbb{R}^n} \sum_{i,j=1,\ldots,n} c_{i,j}(x^i \cdot x^j)$$

$$\text{constrained by} \sum_{i,j=1,\ldots,n} a_{i,j,k}(x^i \cdot x^j) \leq b_k \forall k.$$

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- Uses dot products of vectors.
- Called "Semidefinite" as this may also be defined through positive-semidefinite matrices.

Semidefinite Programming(2)

$$\min_{x^1,\ldots,x^n \in \mathbb{R}^n} \quad \sum_{i,j=1,\ldots,n} c_{i,j}(x^i \cdot x^j)$$

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- There exist efficient algorithms for solving SDPs (up to an arbitrarily small error).

Semidefinite Programming(2)

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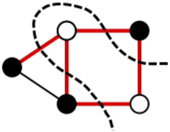
- There exist efficient algorithms for solving SDPs (up to an arbitrarily small error).
- Many problems may be stated as a SDP.

Overview

Maximum Weighted Cut: Problem Statement



Let $G = (V, E)$ be an undirected Graph and let $w : E \to \mathbb{N}$. The goal is to find a Partition $\Pi = (V_1, V_2)$ of V, such that sum of the weights of the edges from V_1 to V_2 is maximized.

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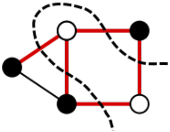

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- MAX-CUT is already NP-Hard.

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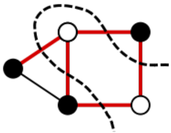

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- MAX-CUT is already NP-Hard.
- Assuming the Unique Games Conjecture, the following algorithm (for MAX-WCUT) is optimal.

Maximum Weighted Cut: Algorithm Outline

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Maximum Weighted Cut: Algorithm Outline

1. Express the problem as a Quadratic Integer Program IQP-CUT.
2. Relax IQP-CUT into a Quadratic Programming Problem QP-CUT for analysis.
3. From QP-CUT derive an equivalent Semidefinite Programming Problem SD-CUT and solve this.
4. Use the solution obtained in the previous step to approximate a solution of the original IQP-CUT.

Maximum Weighted Cut: Constructing IQP-CUT and QP-CUT

$$\text{Maximize} \qquad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - y_i y_j)$$

$$\text{constrained by} \qquad y_i \in \{-1, 1\} \forall i.$$

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Partition by $V_1 = \{y_i | y_i = 1\}$, $V_2 = \{y_i | y_i = -1\}$.
Let $\mathbf{y_i}$ denote a 2-dimensional vector on the unit sphere.

$$\text{Maximize} \quad \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - \mathbf{y_i} \cdot \mathbf{y_j}) \tag{7}$$

$$\text{constrained by} \quad \|y_i\| = 1. \tag{8}$$

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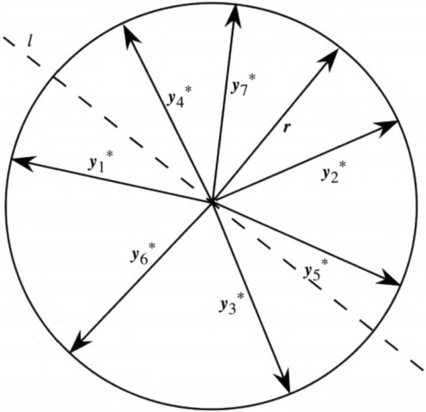
Assume QP-CUT is efficiently solvable.

Maximum Weighted Cut: Randomized Rounding

Randomly choose a straight line through the origin and partition according to the side on which each vector lies on:



Maximum Weighted Cut: Analysis

- The same can be done in higher dimensions, leading to SD-CUT:

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\text{Maximize} \quad & \frac{1}{2} \sum_{(i,j)\in E} w_{ij}(1 - M_{i,j}) \\
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- This can be solved efficiently.
- Straight forward analysis shows that this is an $0.878\ldots-$approximation.

Summary

- Linear Programming and Randomized Rounding are applicable to a wide variety of problems.
- Algorithms are fairly easy to understand.
- Often leads to suprisingly efficient solutions.

Questions?

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Thank you for your attention!