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A Conservative FE-discretisation of the Navier-Stokes-Equation

Thomas Satzger

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Abstract

In this work we present conservative FE-elements which were introduced by Cornelia Blanke in her diploma thesis 2004 [Bla04] .

The usual way to solve a partial differential equation (PDE) with the finite element method (FEM) is to derive the weak formulation and discretize it with the functions in a suitable choosen finite-dimensional function space. For reasons of simplicity one mostley uses linear or bilinear elements for the function space. This choice of the function space isn't always suitable, because the laws of conservation aren't fulfilled. This may lead to instabilities and unphysical results.

In this paper we present this new type of FE-elements which were invented by Cornelia Blanke. With a more physical approach to solve e.g. the Navier-Stokes-Equation with FEM we get the so-called conservative FE-elements. Particularly the laws of conservation for mass and energy in incompressible fluids are fulfilled by construction, which leads to an inherent stability for this method.

1 Motivation

It is common practice for mathematicians to regard the following properties of a numerical method:

- *Consistency*
  This means, that the error of the discretized system tend to zero if the mesh size goes to zero. E.g. consistency means, that the local error tends to zero.
- *Stability*
  It is usual to say that a method is *stable* if an error of the input data has only "small" effects on the solution (the mathematicans would say that there is a continuous relation between the input error and the behaviour of the solution)
- *Convergence*
  A method is said to be *convergent* if the global error of this method tends to zeros if the mesh size tends to zero too. For many problems e.g. for elliptic PDEs holds the fact, that a stable and consistent method is always convergent.
- *Influence of the grid*
  The propertes of a good numerical method, e.g. the convergent rate or smoothness of the solution, should be independent of mesh size and layout.

This point of view is often to narrow minded, because one looks only to the mathematical aspect of a problem. Due to the fact, that the most used PDE have a physical meaning, it would be more appropriate to consider the physical laws of the context.

- *Physical laws*
  Many physical laws describe some kind of conservation. This means, that for instance the energy in a closed system is constant, or that the mass won't vanish or increase. But also momentum is preserved. Unfortunately many kinds of ansatz functions don't have this porperty of conservation.

For instance the common way to practice FEM is to choose a square grid and to take bilinear ansatz functions for the FEM method. We all know that this method is consistent and it converges quite good. But the bilinear ansatz functions don't preserve the energy or the mass. So it can happen that during the computation the mass or the energy in the system increases. But this is totally unrealistic. So you can get unphysical results.
Later on we will present you conservative ansatz function, but first we take a short look at the Navier-Stokes-Equation to understand our problem in a better way.

2 The Navier-Stokes-Equation

In this paper we will discuss about incompressible viscous fluid. This means that the density of the fluid is constant in time and space. Therefore we have three degrees of freedom in our problem:

The pressure p and the velocity u. The velocity u has one component parallel to the $x-$axis and one parallel to the $y-$axis. To be more precise we introduce the following notations:

$$\begin{aligned} u : \Omega \subset \mathbb{R}^2 &\mapsto \mathbb{R}^2 \\ u(t, x1, x2) &:= \begin{pmatrix} u_1(t, x1, x2) \\ u_2(t, x1, x2) \end{pmatrix} =: u(x) \\ p : \Omega \subset \mathbb{R}^2 &\mapsto \mathbb{R} \\ p(t, x1, x2) &=: p(x) \end{aligned}$$

With the abbreviation ∂_t for the derivation $\frac{\partial}{\partial t}$ and ∂_i for $\frac{\partial}{\partial x_i}$ we get the following notations

$$\begin{aligned}
\partial_t u &:= \begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} \\
\nabla \cdot u &:= \operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 \\
\nabla p &:= \begin{pmatrix} \partial_1 p \\ \partial_2 p \end{pmatrix} \\
\Delta u &:= \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} \partial_1^2 u_1 + \partial_2^2 u_1 \\ \partial_1^2 u_2 + \partial_2^2 u_2 \end{pmatrix} \\
(u \cdot \nabla) u &:= \begin{pmatrix} u \cdot \nabla u_1 \\ u \cdot \nabla u_2 \end{pmatrix}
\end{aligned}$$

Now we are able to introduce the Navier-Stokes-Equation. It is:

$$\partial_t u + (u \cdot \nabla) u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p = f \qquad (1)$$

$$\operatorname{div} u = 0 \qquad (2)$$

The equation (1) is called *momentum equation* and (2) is called *continuity equation.*

If we regard the equation (1) we will discover, that this equation deals with accelerations. To get a better view of this we write down an equivalent form of (1):

$$\partial_t u = -(u \cdot \nabla) u + \frac{1}{\mathrm{Re}} \Delta u - \nabla p - f$$

The left hand side gives us the total acceleration of a fluid particle. On the right hand side we have several terms with different meanings.

- $(u \cdot \nabla) u$ is called the *convective* term. This means transport of kinetic energy by moving the fluid particle.
- $\frac{1}{\mathrm{Re}} \Delta u$ is called the *diffusion* term. This is some kind of friction in the fluid. This means that momentum is transported by friction to the particle around itself. Some people also say that this effect is induced by the intermolecular momentum transport.
- ∇p is the *pressure* gradient. It represents the effect of forces which were induced by pressure.
- f is the symbol for the outer forces. This can be for example be gravitational or electromagnetic effects.

The continuity equation (2) says that the mass is preserved if we have an incompressible fluid. It is very useful to make this assumption in many cases, because the ühysics and the computation became much easier. Furthermore even the air can be regarded as incompressible if the velocity is less than $0.3 \cdot$ Mach.

3 Laws of Conservation

Now we will care about the laws of conservation. Especially we take a close look at the energy conservation.

Due to the fact that we regard only with incompressible fluids we can define the energy by

$$E_{kin} = \int_\Omega \frac{1}{2} ||u||^2 \, d\Omega = \int_\Omega \frac{1}{2} u_i u_i \, d\Omega$$

if we use the Einstein Summation.

To be sure that we preserve energy, we must have

$$\frac{d}{dt}E_{kin} \leq 0.$$

How can we see this?
Let us just calculate $\frac{d}{dt}E_{kin}$.

$$\begin{aligned} \frac{d}{dt}E_{kin} &= \frac{d}{dt}\int_\Omega \frac{1}{2}u_iu_i \; d\Omega = \int_\Omega \frac{1}{2}\partial_t(u_iu_i) \; d\Omega \overset{\text{product rule}}{=} \int_\Omega \frac{1}{2}u_i \cdot \partial_t u_i + \partial_t u_i \cdot u_i \; d\Omega \\ &= \int_\Omega u_i \cdot \partial_t u_i \; d\Omega \end{aligned}$$

According to the momentum equation (1) we replace $\partial_t u_i$ by

$$-(u_j\partial_j)u_i + \frac{1}{\mathrm{Re}}\partial_j\partial_j u_i - \partial_i p.$$

In the next step we get

$$\begin{aligned} \frac{d}{dt}E_{kin} &= \int_\Omega u_i \cdot \partial_t u_i \; d\Omega = \int_\Omega u_i \cdot (-(u_j\partial_j)u_i + \frac{1}{\mathrm{Re}}\partial_j\partial_j u_i - \partial_i p) \; d\Omega \\ &= -\int_\Omega u_iu_j\partial_j u_i \; d\Omega + \frac{1}{\mathrm{Re}}\int_\Omega u_i\partial_j\partial_j u_i \; d\Omega - \int_\Omega u_i\partial_i p \; d\Omega \end{aligned}$$

Using the relation $\partial_j(u_ju_i) = u_j \cdot \partial_j u_i + \partial_j u_j \cdot u_i \overset{\partial_j u_j = 0}{=} u_j \cdot \partial_j u_i$ we can simplify several terms. For the first term we get

$$\begin{aligned} 0 &= \oint_{\partial\Omega} u_iuu_i \; d(\partial\Omega) \overset{Gauss}{=} \int_\Omega \partial_j(u_iu_ju_i) \; d\Omega \int_\Omega u_i \cdot \partial_j(u_ju_i) \; d\Omega + \int_\Omega \partial_j u_i \cdot (u_ju_i) \; d\Omega \\ &= \int_\Omega u_iu_j \cdot \partial_j u_i \; d\Omega + \int_\Omega \partial_j u_i \cdot (u_ju_i) \; d\Omega = 2 \cdot \int_\Omega u_iu_j \cdot \partial_j u_i \; d\Omega \end{aligned}$$

In the first "=" sign we used the fact that we regard a closed system which has no net flux. Therefore the first term vanishes.
In an analogous way we get for the last term

$$\begin{aligned} 0 &= \oint_{\partial\Omega} pu \; d(\partial\Omega) \overset{Gauss}{=} \int_\Omega \partial_i(pu_i) \; d\Omega = \int_\Omega p \cdot \partial_i u_i \; d\Omega + \int_\Omega \partial_i p \cdot u_i \; d\Omega \\ &\overset{\partial_i u_i = 0}{=} \int_\Omega \partial_i p \cdot u_i \; d\Omega \end{aligned}$$

Till now we have

$$\begin{aligned} \frac{d}{dt}E_{kin} &= \underbrace{-\int_\Omega u_iu_j\partial_j u_i \; d\Omega}_{=0} + \frac{1}{\mathrm{Re}}\int_\Omega u_i\partial_j\partial_j u_i \; d\Omega - \underbrace{\int_\Omega u_i\partial_i p \; d\Omega}_{=0} \\ &= \frac{1}{\mathrm{Re}}\int_\Omega u_i\partial_j\partial_j u_i \; d\Omega \overset{\text{Green's rule}}{=} -\frac{1}{\mathrm{Re}}\int_\Omega \underbrace{\partial_j u_i \cdot \partial_j u_i}_{\geq 0} \; d\Omega \leq 0. \end{aligned}$$

We have seen that the continuity equation is the key for preserving energy in our system. This leads us to the idea that we should look for an ansatz function f which fulfills pointwise

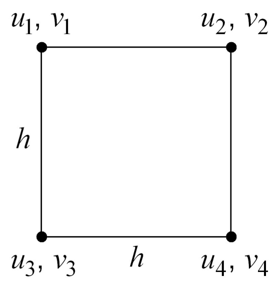
$$\operatorname{div} f = 0.$$

Then we know that the energy is preserved.

4 Conservative FE-elements

First we will state that we use a so called partial staggerd grid. In the figure below we see how the degrees of freedom are orderd. The horizontal velocity is named u and the vertical velocity is named v and the width of the square is h. To say it in words:

- The pressure p is stored in the middle of the square.
- The velocities u and v for x- and y-directions are stored in the corners of the square.



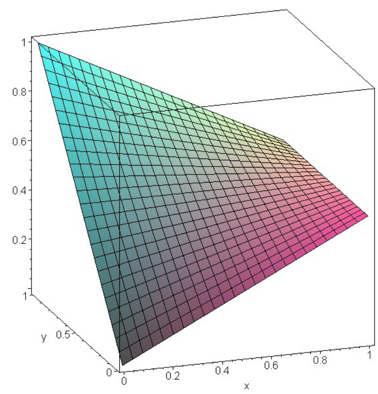
Now we will show that the bilinear ansatz function aren't good because the divergence isn't zero. The bilinear ansatz function is made out of the four elemental shape functions

$$\begin{aligned}
\phi_1(x,y) &= (1-x)\cdot(1-y) \\
\phi_2(x,y) &= x\cdot(1-y) \\
\phi_3(x,y) &= (1-x)\cdot y \\
\phi_4(x,y) &= x\cdot y
\end{aligned}$$

The bilinear interpolation of the velocities $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ reads then

$$\begin{aligned}
\overrightarrow{u} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} &= \begin{pmatrix} u_1(1-x)y + u_2xy + u_3(1-x)(1-y) + u_4x(1-y) \\ v_1(1-x)y + v_2xy + v_3(1-x)(1-y) + v_4x(1-y) \end{pmatrix} \\
&= \begin{pmatrix} u_3 + (u_4 - u_3)x + (u_1 + u_3)y + (-u_1 + u_2 + u_3 - u_4)xy \\ v_3 + (v_4 - v_3)x + (v_1 + v_3)y + (-v_1 + v_2 + v_3 - v_4)xy \end{pmatrix}
\end{aligned}$$

A example of a bilinear function is shown in the figure below.



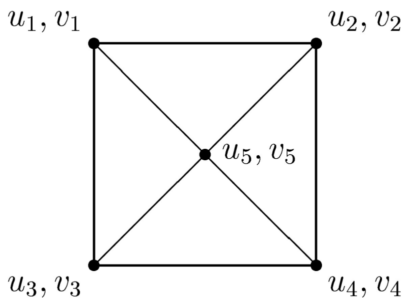
Now we must only apply the div operator and we see that

$$\operatorname{div}\overrightarrow{u} = (u_4 - u_3) + (-u_1 + u_2 + u_3 - u_4)y + (v_1 + v_3) + (-v_1 + v_2 + v_3 - v_4)x \neq 0$$

The reason for this behavior is that the velocities u and v aren't coupled.We leave now this counterexample of a divergentfree FE-element and look ahead to consturct an appropriate ansatz function. In the first step, we derive the *discrete continuity equation*. The idea is that we make a linear interpolation on the edges of the velocitices in the corners. We get then

$$\begin{aligned} & \frac{u_1+u_3}{2}\cdot h + \frac{v_3+v_4}{2}\cdot h = \frac{u_2+u_4}{2}\cdot h + \frac{v_1+v_2}{2}\cdot h \\ \Longleftrightarrow \quad & u_1 - u_2 + u_3 - u_4 - v_1 - v_2 + v_3 + v_4 = 0 \end{aligned}$$

Our task is to fulfill $\operatorname{div} f = 0$ pointwise. The main idea is that we use our freedom in choice for the ansatz function. First we divide the square element into four equal triangels like in the picture below.



On each triangle we make a linear interpolation of the velocities. Now we search a velocity u_5 and v_5 such that on every tringale the divergence is zero. The condition $\operatorname{div} \overrightarrow{u}$ gives use the relations:

- upper triangle

$$\begin{aligned} & \partial_x u = u_2 - u_1, \quad \partial_y v = v_1 + v_2 - 2v_5 \\ & \Longrightarrow v_5 = \frac{1}{2}(v_1 + v_2 + u_2 - u_1) \end{aligned}$$

- lower triangle

$$\begin{aligned} & \partial_x u = u_4 - u_3, \quad \partial_y v = 2v_5 - v_3 - v_4 \\ & \Longrightarrow v_5 = \frac{1}{2}(v_3 + v_4 + u_3 - u_4) \end{aligned}$$

- left triangle

$$\begin{aligned} & \partial_x u = 2u_5 - u_1 - u_3, \quad \partial_y v = v_1 - v_3 \\ & \Longrightarrow u_5 = \frac{1}{2}(u_1 + u_3 + v_3 - v_1) \end{aligned}$$

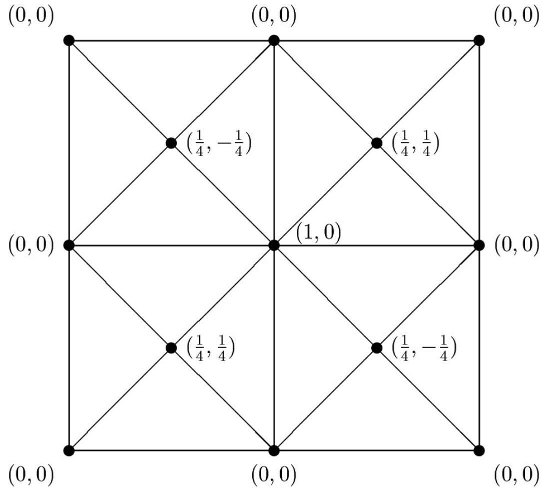
- right triangle

$$\begin{aligned} & \partial_x u = u_2 + u_4 - 2u_5, \quad \partial_y v = v_2 - v_4 \\ & \Longrightarrow u_5 = \frac{1}{2}(u_2 + u_4 + v_2 - v_4) \end{aligned}$$

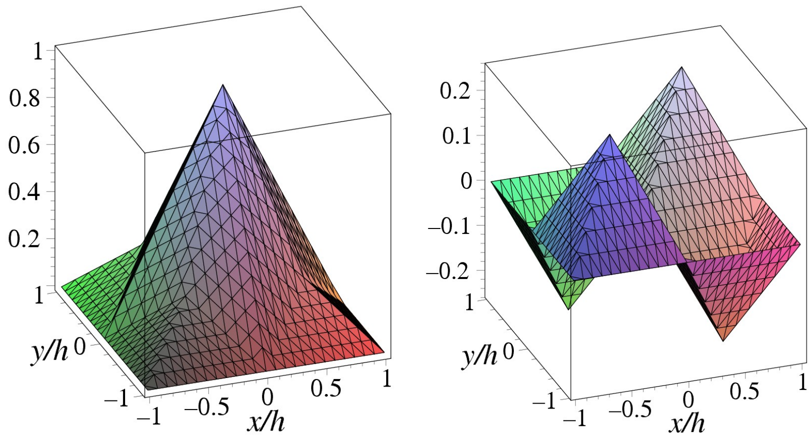
When we regard the discrete continuity equation we see that the first two relations are equivalent and the last to equation are equivalent. The result is that we have a continous function which is divergence free. It is the best, if we define u_5 by the arithmetic mean of the last two equations and analog for v_5.

$$\begin{aligned} u_5 &= \frac{1}{4}(u_1 + u_2 + u_3 + u_4 - v_1 + v_2 + v_3 - v_4) \\ v_5 &= \frac{1}{4}(-u_1 + u_2 + u_3 - u_4 + v_1 + v_2 + v_3 + v_4) \end{aligned}$$

We also see that we get here a coupling of the u and v velocities. The last step isn't so difficult, cause we still have to present a suitable ansatz function. But now we have the right interpolation scheme and we just state the result:



A little bit more illustrative are the following picture of this basis function. You must keep in mind that the u and v component are coupled and therefor both parts are not zero.



People are still working in this field an e.g. in [Wei05] the [Bla04]-elements are agumented there for an better application.

References

[Bla04] C. Blanke. Kontinuitätserhaltende Finite-Elemente-Diskretisierung der Navier-Stokes-Gleichung. *Diplomarbeit, Technische Universität München, Fakultät für Informatik, Forschungs- und Lehreinheit Informatik V*, 2004.

[Wei05] T. Weinzierl. Eine Cache-optimale Implementierung eines Navier-Stokes Lösers unter besonderer Berücksichtigung physikalischer Erhaltungssätze. *Diplomarbeit, Technische Universität München, Fakultät für Informatik, Forschungs- und Lehreinheit Informatik V*, 2005.