An efficient algorithm for stochastic differential equations

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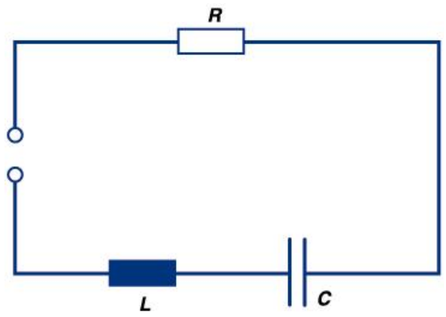
Introduction

The LC-curcuit: a model from the book



- Resistance R
- Induction coil L
- Capacitor C

The corresponding system of equations is

$$\dot{U} = -\frac{I}{C} \qquad\qquad \dot{I} = \frac{U - RI}{L}$$

U is the voltage at the capacitor

I is the current through the coil

The solution is (for small resistance)

$$I(t) = \frac{U_0}{R_0} \cdot \frac{\omega_0}{\omega_e} \cdot e^{-at} \sin(\omega_e t)$$

- Decay constant $a = \frac{R}{2L}$
- Characteristic angular frequency $\omega_0 = \frac{1}{\sqrt{LC}}$
- Angular frequency $\omega_e = \sqrt{|\omega_0^2 - a^2|}$
- Characteristic resistance $R_0 = \sqrt{\frac{L}{C}}$

Observation:

$$C = 40\mu F \pm 5\%$$

Interpretation

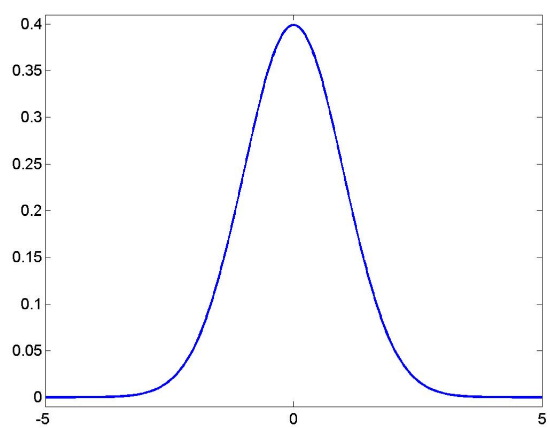
- We cannot be sure about the actual value for the capacity
- Even the given interval $[38, 42]$ is not 100% certain
- Since we cannot measure exactly, the value for the capacity lies in the interval $[C_-, C_+]$ with a likelihood corresponding to that interval

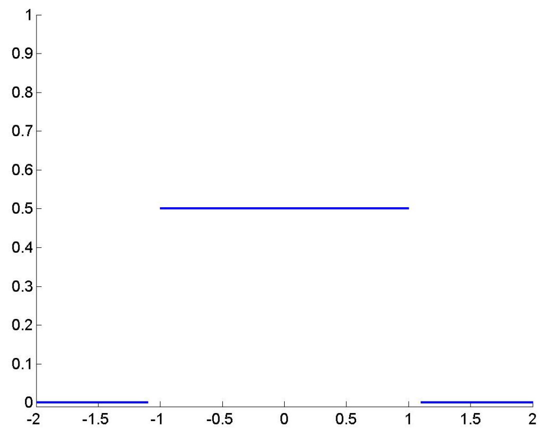
$\Longrightarrow$ Stochastic calculus is required!

Stochastic calculus

Stochastic variables

Variables, which are not given by their value, but by their density function





Gaussian normal distribution (left) and uniform distribution on $[0, 1]$

Moments

- Expectation (first power moment)

$$E[x] = \int x \, \rho(x) dx$$

- Variance (second centred moment)

$$Var[x] = \int (x - E[x])^2 \, \rho(x) dx$$

Functions in stochastic variables

If $f : \mathbb{R} \to \mathbb{R}$ a real function,

$\tau(\theta)$ (real) random variable

$\to \quad f(\tau)$ also (real) random variable

Power moments are given by the density of τ, e.g.

$$E[f(\tau)] = \int_{\mathbb{R}} f(\tau) d\rho(\tau)$$

Important: $E[f(\tau)] \neq f(E[\tau])$

Monte-Carlo Method

Idea: roll the dice for many times, and analyse the result.

E.g. in order to obtain the expected value $E[f(\tau)]$

- Realise τ several times: $t_1, t_2, \ldots, t_N$, so that t_i are approximately ρ-distributed (usually with a random generator)

- Take the N samples $f(t_1), f(t_2), \ldots, f(t_N)$

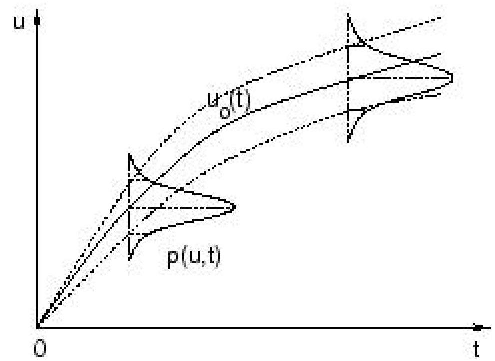
- Calculate the mean $\bar{f} = \frac{1}{N} \sum f(t_i)$.

Thus, Monte-Carlo is a method for calculating integrals.

Problem: N is too large

Stochastic processes

A stochastic process is function of time and chance



A stochastic process $u(t)$ with 3 scenarios, one of them – its expectation $u_0(t)$

Stochastic differential equation

Here:

A differential equation with random input parameters.

Stochastic differential equation

Here:

A differential equation with random input parameters.

Random LC-curcuit

The same equations as in the book model, but the capacity and resistance are normally distributed:

$$\dot{U} = -\frac{I}{C}, \qquad C \sim N[C_0, C_1]$$

$$\dot{I} = \frac{U - RI}{L}, \qquad R \sim N[R_0, R_1]$$

Separation of chance and time

The space of stochastic variables

- Consider Θ the vector space of (real) stochastic variables with expectation 0.

- Θ is a Hilbert space with the inner product

$$< \xi_1, \xi_2 >= E[\xi_1 \xi_2]$$

- Choose an orthogonal basis corresponding to the distribution of input parameters!

Gaussian distribution and Hermite decomposition

Since we assume that the input parameters are normally distributed, we have to choose a system orthogonal with respect to the Gaussian weighting function $w(x) = \exp(\frac{-x^2}{2})$

$\Rightarrow$ Hermite polynomials $H_n(\xi)$, where $\xi \sim N[0,1]$

$$H_0(\xi) = 1, \qquad H_1(\xi) = \xi,$$
$$H_2(\xi) = \xi^2 - 1, \qquad H_3(\xi) = \xi^3 - 3\xi, \ \ldots$$

Project a stochastic process on the Hermite polynomials!

$$x(t,\theta) = \sum_{i=0}^{\infty} u_i(t) H_i(\xi(\theta))$$

The coefficients u_i are now deterministic functions of time! They are given by

$$u_i(t) = \frac{1}{E[H_i^2(\xi)]} E[H_i(\xi)\, x(t,\theta)]$$

$\longrightarrow$ Another form of the Fourier decomposition

Application to differential equations

Consider the ODE

$$\dot{x} = f(x, t, \theta)$$

with appropriate initial conditions. Its solution is the stochastic process $x(t, \theta)$.

- Write x as

$$x(t,\theta) \approx \sum_{i=0}^{P} u_i(t) H_i(\xi)$$

- Plug in:

$$\sum_{i=0}^{P} \dot{u}_i(t) H_i(\xi) = f(\sum_{i=0}^{P} u_i(t) H_i(\xi), t, \theta)$$

- Galerkin condition yields

$$\left(\sum_{i=0}^{P} \dot{u}_i(t) H_i(\xi) - f(\sum_{i=0}^{P} u_i(t) H_i(\xi), t, \theta) \right) \perp H_k(\xi), \;\; k = 0, \ldots, P$$

- This results in the system

$$\dot{u}_k = E[H_k^2(\xi)]\, E\left[H_k(\xi)\, f\left(\sum_{i=0}^{P} u_i(t) H_i(\xi), t, \theta\right)\right]$$

Stochastic LC-curcuit

Assume that R, L, C are functions of ξ, $\xi \sim N[0,1]$. Then the approach is

$$U(t,\theta) \approx \sum_{i=0}^{P} u_i(t) H_i(\xi), \qquad I(t,\theta) \approx \sum_{i=0}^{P} v_i(t) H_i(\xi).$$

Doing the same steps as before we obtain

$$\|H_k(\xi)\|^2\,\dot{u}_k = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} -H_k(\xi)\frac{1}{C(\xi)}\sum_{i=0}^{P} v_i(t)H_i(\xi)\,e^{-\frac{\xi^2}{2}}d\xi$$

$$\|H_k(\xi)\|^2\,\dot{v}_k = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} H_k(\xi)\frac{\sum_{i=0}^{P} u_i(t)H_i(\xi) - R(\xi)\sum_{i=0}^{P} v_i(t)H_i(\xi)}{L(\xi)}\,e^{-\frac{\xi^2}{2}}d\xi$$

with initial conditions $u_0(0) = U_0$, $v_0(0) = I_0$, $u_i(0) = v_i(0) = 0$ for $i > 0$.

Comparison with Monte-Carlo

- Decay constant $a = \frac{R}{2L}$
- Characteristic angular frequency $\omega_0 = \frac{1}{\sqrt{LC}}$
- Angular frequency $\omega_e = \sqrt{|\omega_0^2 - a^2|}$
- Characteristic resistance $R_0 = \sqrt{\frac{L}{C}}$

Let the capacity be normally distributed !

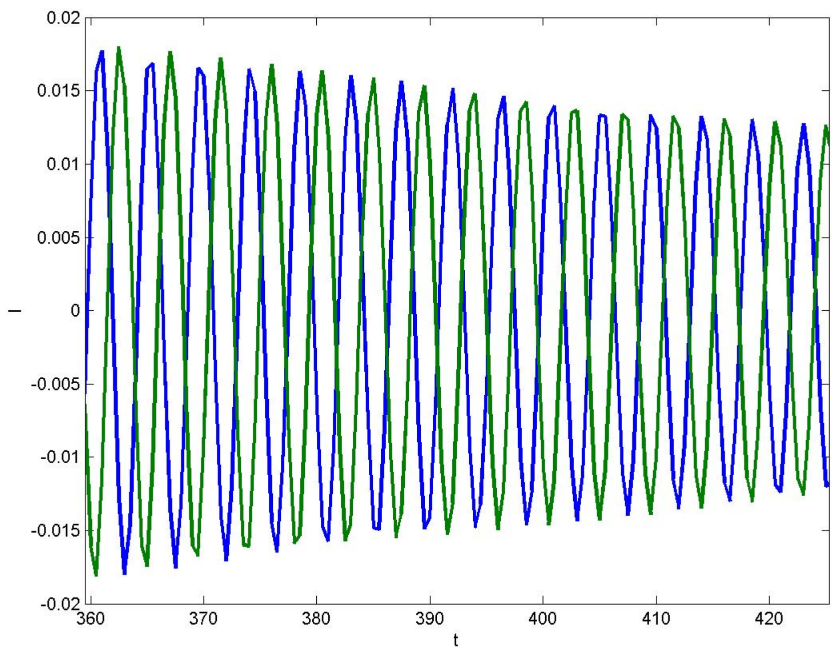
In terms of Hermite decompostion this means

$$C = C_0 + C_1\xi$$

where C_0 is the expected value and C_1 is the standard deviation

Intuition:

- A change in frequency
- No change in damping



0.02
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
-0.02
I
360
370
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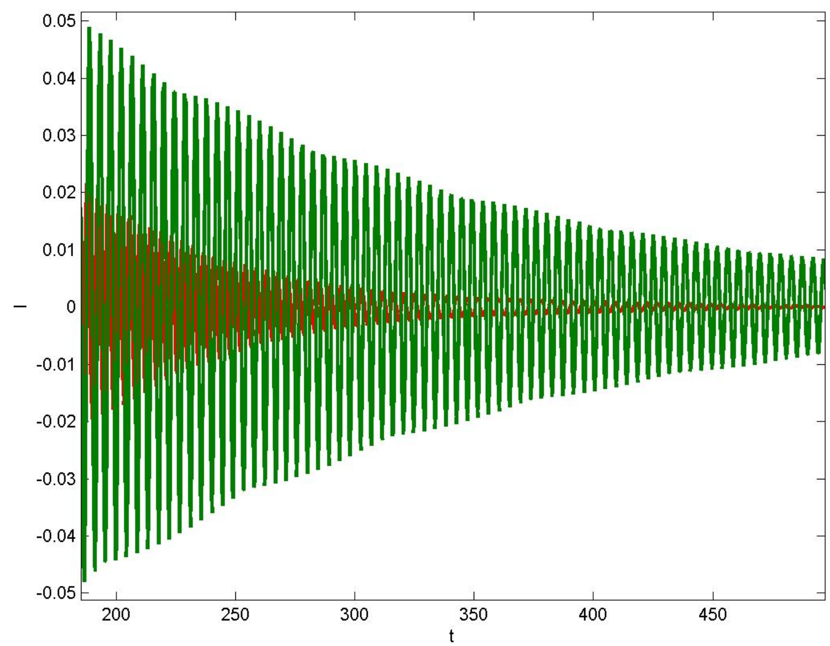
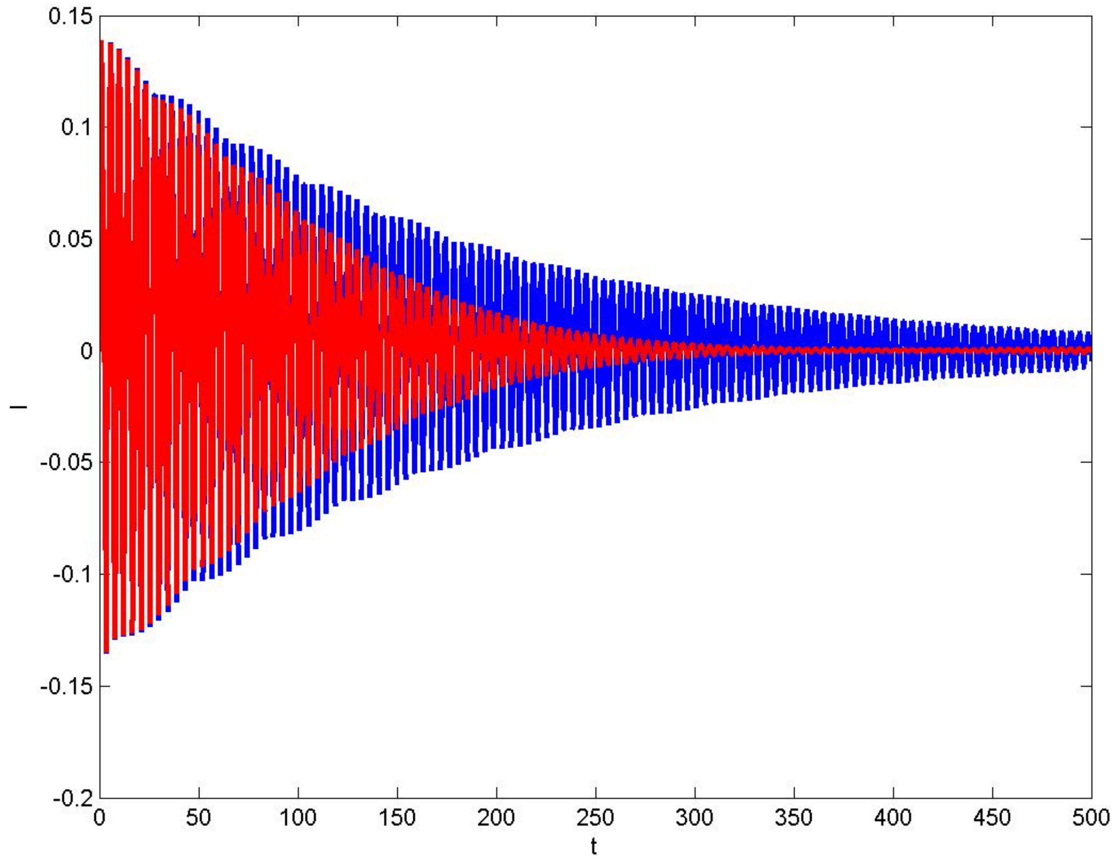
390
400
410
420
t



The stochastic LC-equations solved with Hermite decomposition (left) and with Monte-Carlo, 10000 samples (right)

Letting L vary we obtain



Conclusions:

- Physical systems may show a behaviour different from the one predicted by deterministic models

- For stochastic differential equations spectral methods such as Hermite decompostion are often an efficient approach, competing with Monte-Carlo in accuracy