

Sparse Grids – Handout

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Course 2: Numerical Simulation – From Models to Visualizations

1 Introduction

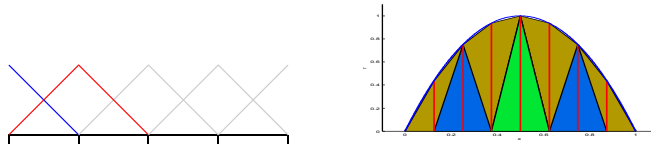
- We want to find a function $u : \Omega \rightarrow \mathbb{R} \in V$ with special properties
- We can only find an approximate solution $u_S \in S \subset V$ with $\dim(S)$ finite
- With $S = \text{span}\{\phi_i : 1 \leq i \leq n\}$: linear combination $u_S = \sum_{i=1}^n \alpha_i \cdot \phi_i$

2 Hierarchical Basis

2.1 In 1 dimension

piecewise linear approach in $\Omega[0, 1]$ with hat functions as basis

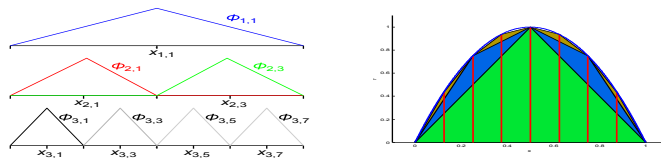
- with the nodal point basis:



$$u_\ell = \sum_{i=1}^{2^\ell - 1} \alpha_i \cdot \phi_{\ell,i}, \quad \alpha_i = u(x_{\ell,i})$$

problem: the coefficients remain large for increasing ℓ

- and with the *hierarchical basis*:



hierarchical increments: $W_\ell := \text{span}\{\phi_{\ell,i} : i \in I_\ell\}$, with $I_\ell := \{i : 1 \leq i \leq 2^\ell - 1, i \text{ odd}\}$

$$\Psi_\ell := \bigcup_{k=1}^{\ell} \bigcup_{i \in I_k} \phi_{k,i}, \quad \text{span}(\Psi_\ell) = S_\ell := \bigoplus_{k=1}^{\ell} W_k$$

hierarchical surpluses: $w_\ell := \sum_{i \in I_\ell} v_{\ell,i} \cdot \phi_{\ell,i} \in W_\ell$, $u(x) = \sum_{\ell=1}^{\infty} w_\ell(x)$

$$v_{\ell,i} = u(x_{\ell,i}) - \frac{u(x_{\ell,i-1}) + u(x_{\ell,i+1})}{2}$$

- norms of the hierarchical surpluses ($h_\ell := 2^{-\ell}$):

$$\|w_\ell\|_\infty \leq \frac{h_\ell^2}{2} \|u''\|_\infty, \quad \|w_\ell\|_2 \leq \frac{h_\ell^2}{3} \|u''\|_2, \quad \|w_\ell\|_E \leq \frac{h_\ell}{2} \|u''\|_\infty$$

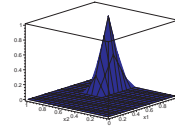
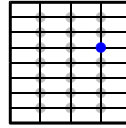
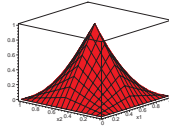
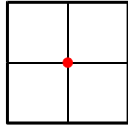
- approximation error in 1 dimension ($n = \ell$):

$$\|u - u_n\|_\infty = \|u - u_n\|_2 = \mathcal{O}(h_n^2), \quad \|u - u_n\|_E = \mathcal{O}(h_n)$$

2.2 In 2 or more dimensions

- now $u(x) = u(x_1, \dots, x_d)$ and ℓ, h_ℓ become vectors
- product of hat functions as basis functions $\rightarrow d$ -linear functions:

$$\phi_{\ell,i}(x) = \prod_{j=1}^d \phi_{\ell_j, i_j}(x_j)$$



$\phi_{(1,1),(1,1)}$ and $\phi_{(2,3),(3,5)}$

- the *hierarchical surpluses* in 2 or more dimensions:

$$\|w_\ell\|_\infty \leq 2^{-d} \cdot 2^{-2|\ell|_1} \cdot \|u''\|_\infty, \quad \|w_\ell\|_2 \leq 3^{-d} \cdot 2^{-2|\ell|_1} \cdot \|u''\|_2$$

$$\|w_\ell\|_\infty \leq \frac{1}{2 \cdot 12^{(d-1)/2}} \cdot 2^{-2|\ell|_1} \cdot \left(\sum_{j=1}^d 2^{2\ell_j} \right)^{\frac{1}{2}} \cdot \|u''\|_\infty$$

- For $d > 1$ it is no longer clear, which sets of W_ℓ are good for approximation

2.3 Sparse grids

What is a “good” approximation: Look at the cost-benefit ratio of each ℓ

- costs: number of points $\hat{=}$ functions in W_ℓ

$$c_\infty(\ell) = c_2(\ell) := |I_\ell| = |\{1 \leq i \leq 2^\ell - 1, \text{ all } i_j \text{ odd}\}| = 2^{|\ell|_1 - d}$$

- benefits: the bound of each $\|w_\ell\|$

$$b_\infty(\ell) = b_2(\ell) := 2^{-2|\ell|_1}$$

- cost-benefit-ratio:

$$cbr(\ell) = \frac{b(\ell)}{c(\ell)}$$

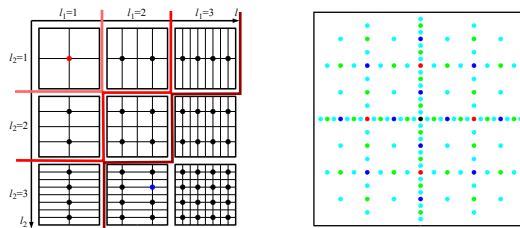
- take only those $\ell \in \mathbb{N}^d$ which have good cost-benefit ratios

$$cbr_2(\ell) = cbr_\infty := \frac{b_*(\ell)}{c_*(\ell)} = 2^{-3|\ell|_1+d}$$

\Rightarrow the good ℓ are those where $|\ell|_1$ is small

$$L_n^1 := \{\ell \in \mathbb{N}^d : |\ell|_1 \leq n + d - 1\}, \quad S_n^1 := \bigoplus_{\ell \in L_n^1} W_\ell$$

- We get a diagonal cut in the figure of the hierarchical increments:



- The so constructed grid is called a *sparse grid*
- Comparison to full grids:

$$L_n^\infty := \{\ell \in \mathbb{N}^d : |\ell|_\infty \leq n\}, \quad S_n^\infty := \bigoplus_{\ell \in L_n^\infty} W_\ell$$

$$\begin{aligned} \dim(S_n^\infty) &= \mathcal{O}(2^{nd}) & \|u - u_n^\infty\|_{2,\infty} &= \mathcal{O}(2^{-2n}) & \|u - u_n^\infty\|_E &= \mathcal{O}(2^{-n}) \\ \dim(S_n^1) &= \mathcal{O}(2^n \cdot n^{d-1}) & \|u - u_n^1\|_{2,\infty} &= \mathcal{O}(2^{-2n} \cdot n^{d-1}) & \|u - u_n^1\|_E &= \mathcal{O}(2^{-n}) \end{aligned}$$

3 Conclusion

- sparse grids are very good, especially when dealing with high dimensional problems
- when having $\Omega \neq [0, 1]^d$: transformation or approximation with cuboids
- others than linear approaches are possible: e.g. polynomials, wavelets, etc.
- applications of sparse grids are e.g. numerical quadrature, solving PDEs, data-mining, etc.

References

- [1] H.-J. Bungartz, M. Griebel – Sparse grids, *Acta Numerica*, pp. 147-269, 2004
- [2] M. Bader, S. Zimmer – lecture’s slides “Algorithmen des Wissenschaftlichen Rechnens”, *TU München*, summer term 2005 (<http://www5.in.tum.de/lehre/vorlesungen/algowiss/ss05/material.html>)