PART I. Basic Concepts

1.1 Introduction

1.2 Basic Terminology of Structural Vibration

1.2.1 Common Vibration Sources

1.2.2 Forms of Vibration

1.3 Structural Vibration

1.2 Basic Terminology of Structural Vibration

- The term vibration describes repetitive motion that can be measured and observed in a structure
- Unwanted vibration can cause fatigue or degrade the performance of the structure. Therefore it is desirable to eliminate or reduce the effects of vibration
- In other cases, vibration is unavoidable or even desirable. In this case, the goal may be to understand the effect on the structure, or to control or modify the vibration, or to isolate it from the structure and minimize struc-tural response.

1.2.1. Common Vibration Sources

- Industrial operations such as pressing or forging are common sources of ground-borne vibration
- It can be critically important to ensure that the natural frequencies of the structural system do not match the operating frequencies of the equipment
- The dynamic amplification which can occur if the frequencies do coincide can lead to hazardous structural fatigue situations, or shorten the life of the equipment.

1.2.1. Common Vibration Sources

Common vibration sources

- External sources
  - Seismic activity
  - Subway, road and rail systems
  - Industrial metalworking facilities
  - Construction equipments
  - Wind
- Internal sources
  - Elevator and conveyance systems
  - Fluid pumping equipment
  - Aerobics and exercise rooms

1.2.2 Forms of Vibration

- **Free vibration** is the natural response of a structure to some impact or displacement
- **Forced vibration** is the response of a structure to a repetitive forcing function that causes the structure to vibrate at the frequency of the excitation
- **Sinusoidal vibration** is a special class of vibration. The structure is excited by a forcing function that is a pure tone with a single frequency
- **Random vibration** is very common in nature. The vibration you feel when driving a car result from a complex combination of the rough road surface, engine vibration, wind buffeting the car's exterior, etc

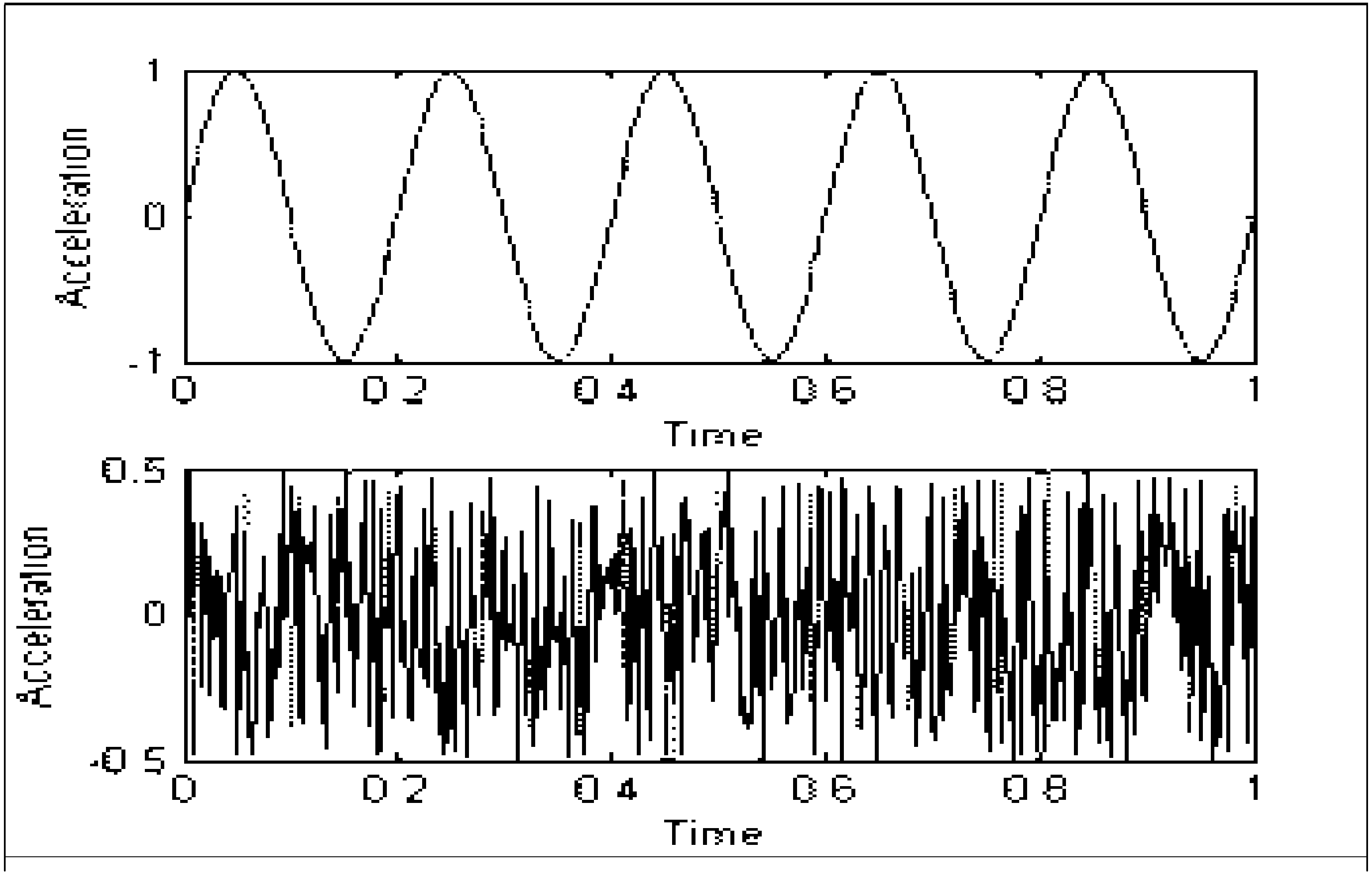
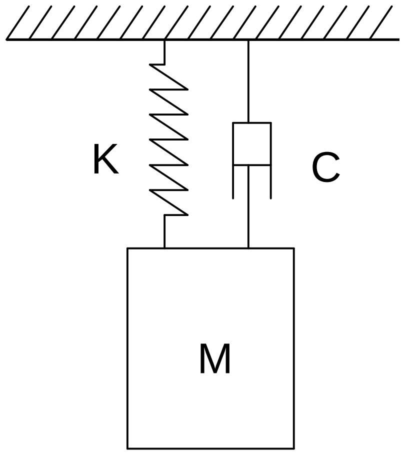


Figure 1.2.Sinusoidal vibration (top) and random vibration (bottom)

1.3 Structural Vibration



- The simplest vibration model is the single-degree-of-freedom, or mass-spring-damper model
- It consists of a sim-ple mass (M) that is suspended by an ideal spring with a known stiffness (K) and a dashpot damper from a fixed support
- If you displace the mass by pulling it down and releasing it, the mass will respond with motion similar to Fig. 1. 4

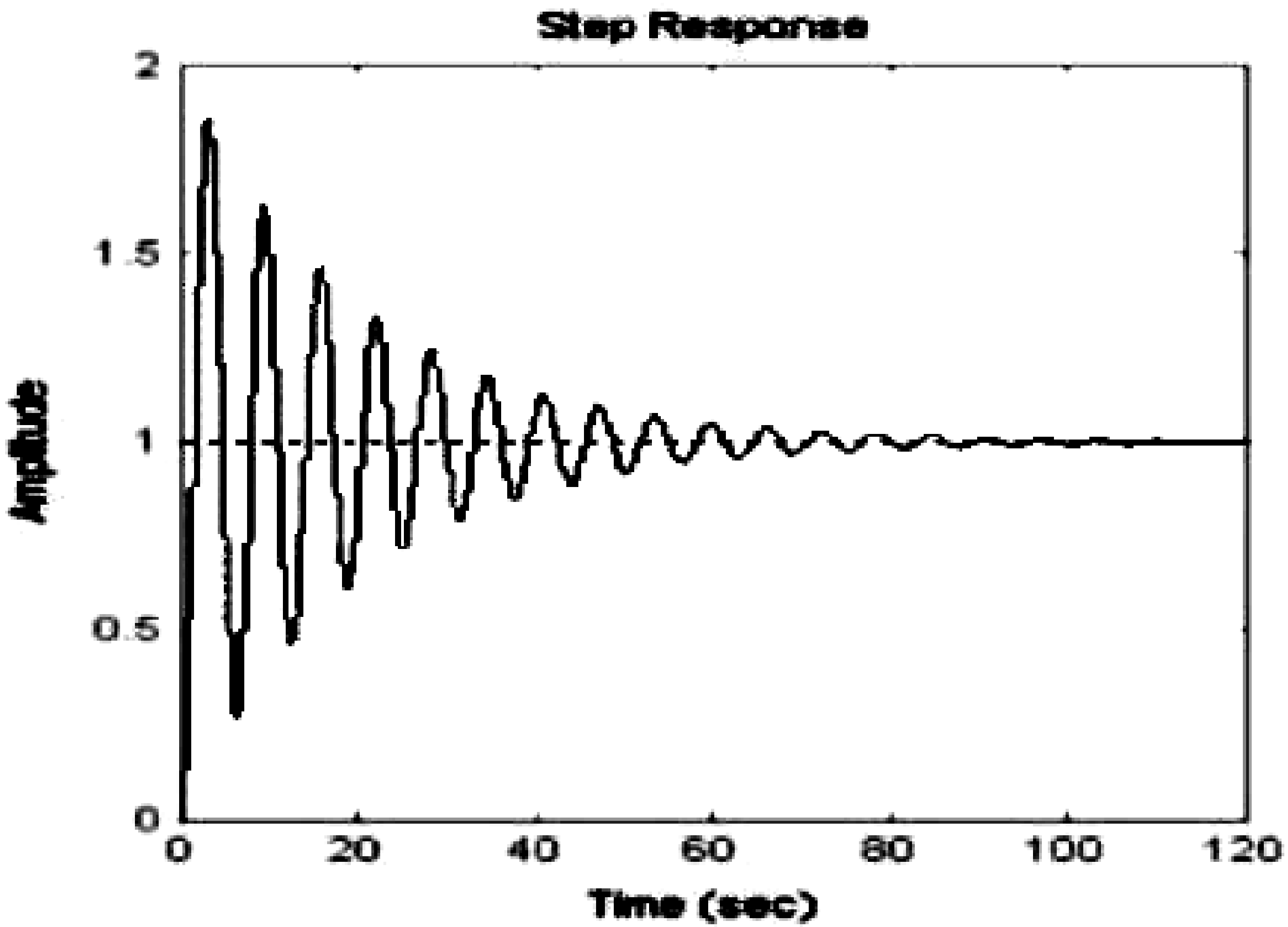


Figure 1.4. Free Vibration of Mass-Spring-Damper Model

1.3 Structural Vibration

- Frequency is measured in cycles per second with units of Hertz (Hz). Assuming the damping is small, then the mathematical relationship is given by
- $$w_n = \sqrt{\frac{K}{M}} \qquad (1)$$
- A larger stiffness will result in a higher • $_n$, and a larger mass will result in a lower • $_n$.
- Usually damping is described in terms of the damping ratio--˜. That ratio is related to C by
- $$z = \frac{C}{2Mw_n} \qquad (2)$$

Modal shapes

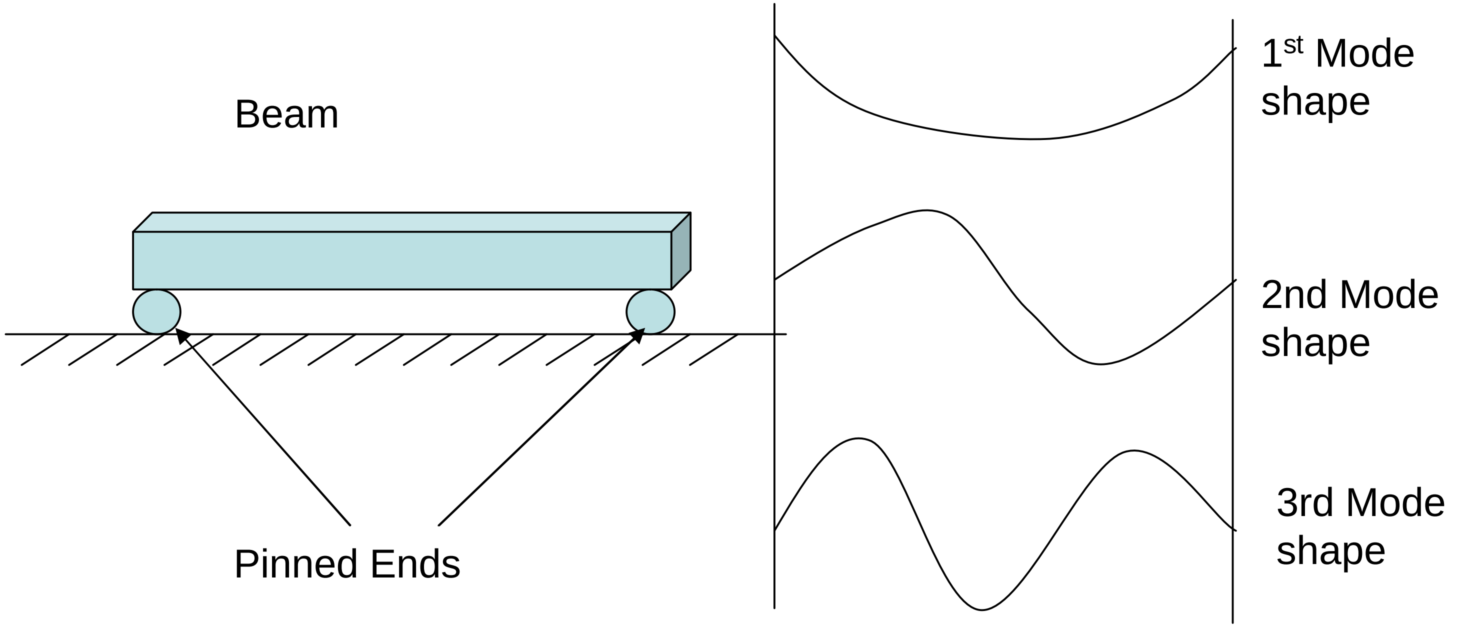


Fig.1. 5 A beam fixed at both ends.

- Each mode shape corresponds to a peak frequency at which all points move in unison
- The higher modes are harder to detect and have less effect on the overall vibration of the structure

PART II. Vibration of Plates

2.1 General Theory

2.2 Transverse Vibration of a rectangular plate.

2.1 General Theory

In three-dimensional elasticity theory the stress at a point is specified by the six quantities:

$s_x, s_y, s_z -$ the components of direct stress;

$s_{xe}, s_{yz}, s_{zx} -$ the components of shear stress.

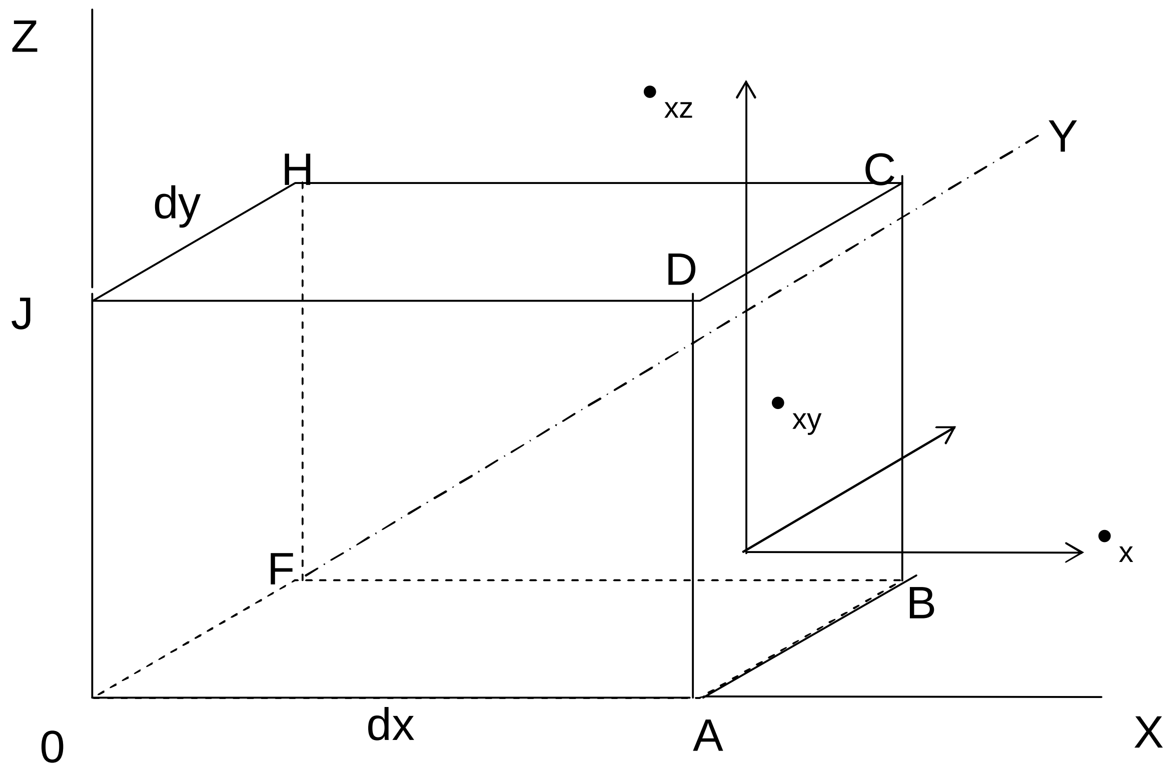


Fig. 2.1. Stress components on the face ABCD of an element

The components of direct stress and strain are related by Hooke's law, extended to include Poisson's ratio effects,

$$e_x = \left({}^{1}\!/_{E} \right) \left[s_x - v (s_y + s_z) \right] \qquad (3)$$

with analogous expressions for $e_y, and\, e_z$, The components of shear stress and strain are related by

$$e_{xy} = \left({}^{1}\!/_{G} \right) s_{xy}, \text{ etc} \qquad (4)$$

In these equations E, G and v are the elastic constants, Young's modulus, shear modulus (or modulus of rigidity) and Poisson's ratio, respectively.

$$E = 2G(1+v) \qquad (5)$$

$$e_x = \frac{\text{Increase in length}}{\text{Initial length of}}$$

$$= \frac{[u + (\partial u / \partial x) dx] - dx}{dx}$$

$$= \partial u / \partial x \qquad (6)$$

Similarly, $e_y = \partial v / \partial y$ and

$e_z = \partial w / \partial z$

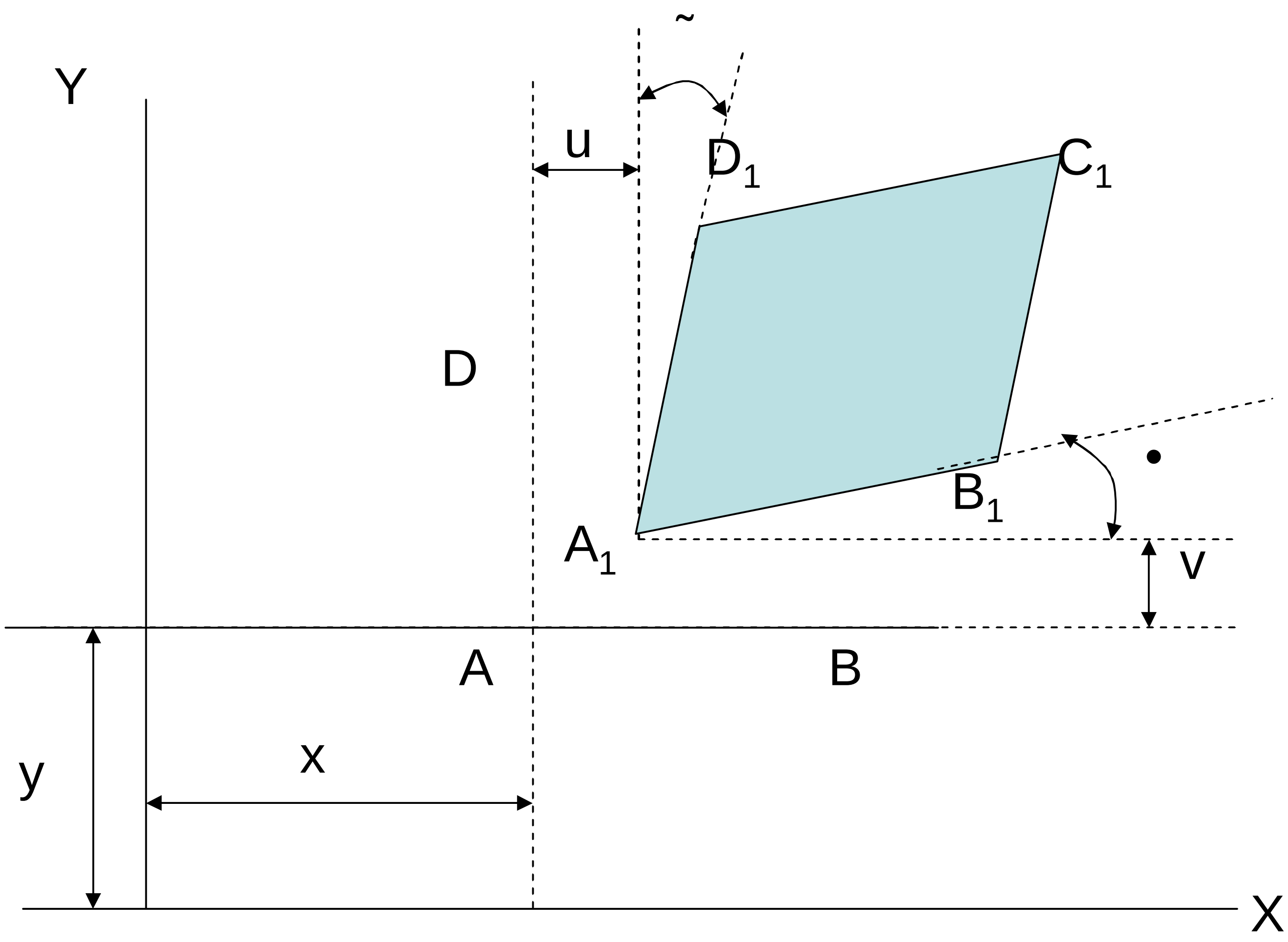


Fig. 2.2. Displacement in the plane OXY

The deformed shape of the element is the parallelogram A_1 B_1 C_1 D_1 and the shear strain is (a + •). Thus for small angles

$$e_{xy} = a + b = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (7)$$

Similarly,

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad \text{and} \quad e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

For an element of sides dx , dy and dz (Fig. 5.1) the force due to the stress s_x *is* $s_x dydz$ *;* if the corresponding strain is e_x, the extension in the X-direction is $e_x dx$ and the work done on the element is

$$\frac{1}{2}(s_x dydz)(e_x dx) = \frac{1}{2} s_x e_x dV \qquad (8)$$

Using, $e_{xy} = a + b$ *,* the work done on the element is

$$\frac{1}{2} s_{xy} e_{xy} dV$$

Generalizing for a three-dimensional state of stress, the strain energy in an elastic body

$$d = \int_V \frac{1}{2}(s_x e_x + s_y e_y + s_z e_z + s_{xy} e_{xy} + s_{yz} e_{yz} + s_{zx} e_{xz}) dV \qquad (9)$$

2.2 Transverse Vibrations of Rectangular Plate

Consider the plate element shown below Fig.(2.5).

1.The plate is thin and of uniform thickness *h;* thus the free surfaces of the plate are the planes

$$z = \pm \frac{1}{2} h$$

2.The direct stress in the transverse direction s_z is zero. This stress must be zero at the free surfaces, and provided that the plate is thin, it is reasonable to assume that it is zero at any section z.

- Stresses in the middle plane of the plate (membrane stresses) are neglected, i.e. transverse forces are supported by bending stresses, as in flexure of a beam.
- Plane sections that are initially normal to the middle plane remain plane and normal to it. Thus with this assumption the shear strains s_{xz} and s_{yz} are zero.
- Only the transverse displacement *w* (in the Z-direction) has to be considered.

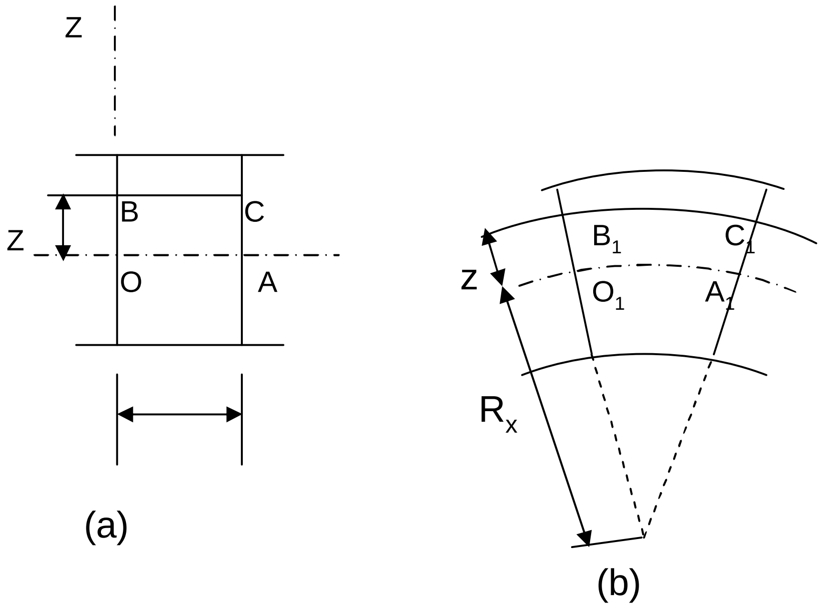


Fig.2.3. Element of plate.
(a) Unstrained. (b) Strained

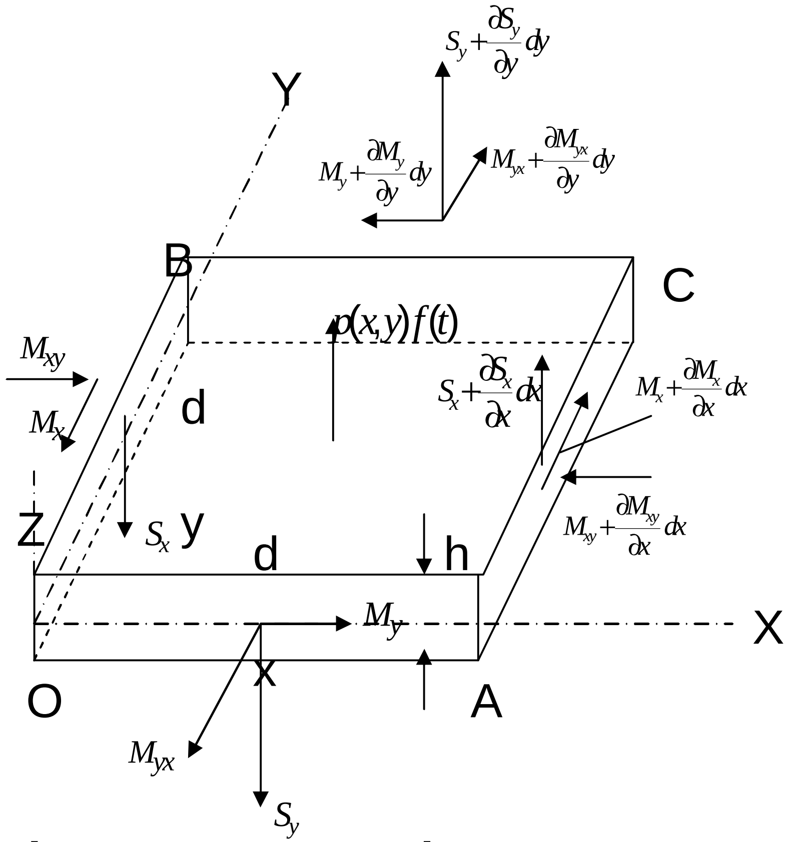


Fig.2.5 Forces and moments on an element of a plate

If *OA* represents the middle surface of the plate, then *OA* = ˜ $_1$ • $_1$ from the third assumption; also ˜ $_1$ • $_1$ = $R_x d$q , where R_x is the radius of curvature of the deformed middle surface. Thus the strain in *BC,* at a distance z from the middle surface, Thus the strain in *BC,* at a distance z from the middle surface,

$$e_x = \frac{B_1C_1 - BC}{BC} = \frac{\left(R_x + z\right) - R_x}{R_x} = \frac{z}{R_x}$$

The relation between the curvature and the displacement of the middle surface, w, is:

$$\frac{1}{R_x} = -\frac{\partial^2 w}{\partial x^2}$$

Thus

$$e_x = -z\frac{\partial^2 w}{\partial x^2} \qquad (10)$$

Similarly

$$e_y = -z\frac{\partial^2 w}{\partial y^2}$$

Thus the strain in *BC,* at a distance z from the middle surface,

$$e_x = \frac{B_1C_1 - BC}{BC} = \frac{(R_x + z) - R_x}{R_x} = \frac{z}{R_x}$$

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$$\frac{1}{R_x} = -\frac{\partial^2 w}{\partial x^2} \quad \text{Thus} \quad e_x = -z\frac{\partial^2 w}{\partial x^2} \tag{10}$$

Similarly $$e_y = -z\frac{\partial^2 w}{\partial y^2}$$

$$\left[\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)\right],$$

where u and v are the displacements at depth z in the *X*- and Y-directions, respectively. Using the assumption hat sections normal to the middle plane remain normal to it,

$$u = -z\frac{\partial w}{\partial x}\ .$$ Similarly $$v = -z\frac{\partial w}{\partial y}$$

Thus

$$e_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{11}$$

(In equation (11) the term $\frac{\partial^2 w}{\partial x \partial y}$ is the twist of the surface.)

From the second assumption and equations (3) and (4), the stress-strain relations for a thin plate are

$$\left.\begin{aligned} s_x &= \frac{E}{1-v^2}\left(e_x + v e_y\right) \\ s_y &= \frac{E}{1-v^2}\left(e_y + v e_x\right) \\ &\text{Also} \\ s_{xy} &= \frac{E}{2(1-v)} e_{xy} \end{aligned}\right\} \quad (12)$$

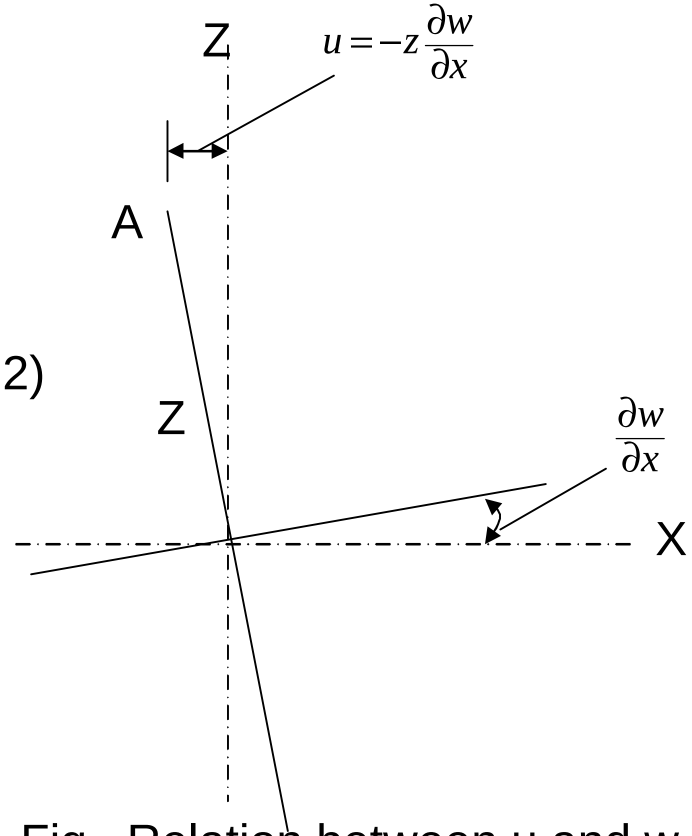

Fig. Relation between u and w

Also $\quad s_{xy} = \dfrac{E}{2(1-v)} e_{xy}$

Substituting from equations (10), (11) and (12) in the strain energy expression (9),

$$d = \int_0^a \int_0^b \int_{-h/2}^{h/2} \frac{E}{2(1-v^2)} \left[e^2{}_x + e^2{}_y + 2v e_x e_y + \frac{1}{2}(1-v^2) e^2{}_x e_y \right] dz\,dy\,dx$$

$$= \frac{D}{2} \int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2v \left(\frac{\partial^2 w}{\partial x^2} \right)\left(\frac{\partial^2 w}{\partial y^2} \right) + 2(1-v) \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] dy\,dx \qquad (13)$$

after integrating with respect to z, where

$$D = \frac{Eh^3}{12(1-v^2)}$$

If • is the density of the plate, the kinetic energy

$$\Im = \int_0^a \int_0^b \int_{-h/2}^{h/2} \frac{1}{2} \mathrm{r} \left(\frac{\partial w}{\partial t} \right)^2 dz dy dx$$

$$= \frac{1}{2} \mathrm{r}\, h \int_0^a \int_0^b \left(\frac{\partial w}{\partial t} \right)^2 dy dx \qquad (14)$$

after integrating with respect to z. Expressions (13) and (14) will be used with the finite element methods.

the moments shown in Fig. 2.5, we have

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz \qquad M_y = \int_{-h/2}^{h/2} \sigma_y z dz$$

;

and

$$M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \sigma_{xy} z dz \tag{15}$$

The equilibrium equations, obtained by resolving in the Z-direction and taking moments about the *Y*- and X-axes, are, after dividing by *dx dy:*

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + p(x, y)\, f(t) = \mathrm{r}\, h \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - S_x = 0 \quad \text{and} \quad -\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + S_y = 0$$

Eliminating S_x and S_y

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x, y)\, f(t) = \mathrm{r}\, h \frac{\partial^2 w}{\partial t^2}$$

(16)

Substituting from equations (10) to (12) in equations (15) and integrating with respect to z.

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right)$$

,

and

$$Mxy = -D(1-v)\frac{\partial^2 w}{\partial x \partial y} \qquad (17)$$

Substituting from equations (17) in equation (16) gives the equilibrium equation for an element of the plate in terms of

w and its derivatives,

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + \text{r}\, h\frac{\partial^2 w}{\partial t^2} = p(x, y)\, f(t) \qquad \textbf{(18)}$$

PART III. Application of FEM

3.1 Finite Element Method (FEM)

3.2 Element Stiffness and Mass Matrices

3.2.1 Axial Element

3.3 FEM for In-plane Vibration

3.4 FEM for Transverse Vibration.

3.1 Finite Element Method

In static analysis. The finite element method solves equations of the form

$$[K]\{U\}=\{P\} \tag{19}$$

where [K] is a global stiffness matrix, {U} is a vector of nodal displacement, and {P} is a vector of nodal forces. The dynamic analysis usually starts with an equation of the form

$$\left([K]-w^2[M]\right)\{U\}=\{P\} \tag{20}$$

where • in this equation denotes the frequencies of vibration and [M] is the mass matrix

For a free vibration, we have {P} = 0 and Eq. 20. reduces to the following expression:

$$\left(\left[K\right]-\mathrm{w}^{2}\left[M\right]\right)\left\{U\right\}=\left\{0\right\} \qquad (21)$$

For a nontrivial solution, we have

$$\left[K\right]=\mathrm{w}^{2}\left[M\right] \qquad (22)$$

which represents the generalized Eigenvalue problems.

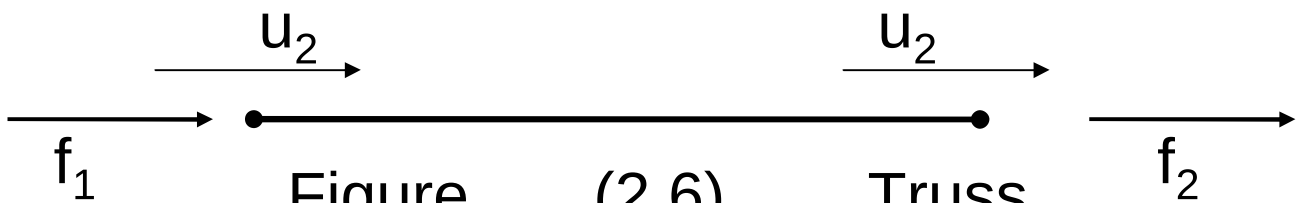

Figure (2.6). Truss element.

$$u(x) = b_1 + b_2 x \tag{23}$$

where b_1 and b_2 are constants. In matrix form Equation (23) may be written as follows:

$$u(x) = \{1 \quad x\} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{24}$$

By using Eq. (23) and applying the boundary conditions $u(0) = u_1$ and $u(L) = u_2$, where L is the length of the elements, we find

$$u(0) = b_1 + (0)b_2 \qquad (25)$$

$$u(L) = b_2 + Lb_2 \qquad (26)$$

Equations (25) and (26) are written in matrix form as follows

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \qquad (27)$$

The same two equations may also be written in a matrix form that relates b_1 and b_2 and u_1 and u_2 in the following way

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (28)$$

By substituting Eq. (28) into Eq. (24), we find

$$u(x) = \begin{Bmatrix} 1 - \dfrac{x}{L} & \dfrac{x}{L} \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (29)$$

or, in general

$$u(x) = \begin{Bmatrix} H_1 & H_2 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (30)$$

where H, in Eq. (30) are referred to as shape functions or interpolation functions

The axial strain ε in Fig 2.7 is the rate change of $u(x)$ with respect to x

. Therefore, by differentiating Eq. (29) with respect to x, we find

$$\varepsilon = \left\{ -\frac{1}{L} \quad \frac{1}{L} \right\} \left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\} \qquad (30)$$

By applying Hooke's law, the normal stress ˜ at cross sections along the length of the element is given by the expression

$$\sigma = E\varepsilon \qquad (31)$$

where • is Young's modulus of elasticity and ˜ is given by Eq. (31).

The total energy Π stored in the element is defined by

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^T\left(\frac{du}{dx}\right)dx - f_1u_1 - f_2u_2 \qquad (33)$$

where A is the cross-sectional area of the member. By using Eq. (30),

the expression given by Eq. (33) may be written as follows:

$$\Pi = \frac{1}{2}\int_0^L EA\{u_1 \quad u_2\}\begin{pmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{pmatrix}\begin{pmatrix} -\frac{1}{L} & \frac{1}{L} \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}dx - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (34)$$

When • A is constant, we have

$$\int_0^L dx = L \qquad (35)$$

and equation 34. yields

$$\Pi = \frac{EAL}{2}\{u_1 \quad u_2\}\begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2}\{u_1 \quad u_2\}\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \{f_1 \quad f_2\}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (36)$$

For a stationary condition, we have

$$\frac{\partial \Pi}{\partial u_i} = 0 \qquad (37)$$

On this basis, use of Eq. (37) yields

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \qquad (38)$$

By comparing Eq. (38) with Eq. (19), we conclude that the stiffness matrix [K] of the element is

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (39)$$

3.3 Finite Element Method for In-plane Vibration of Plate

- A plate

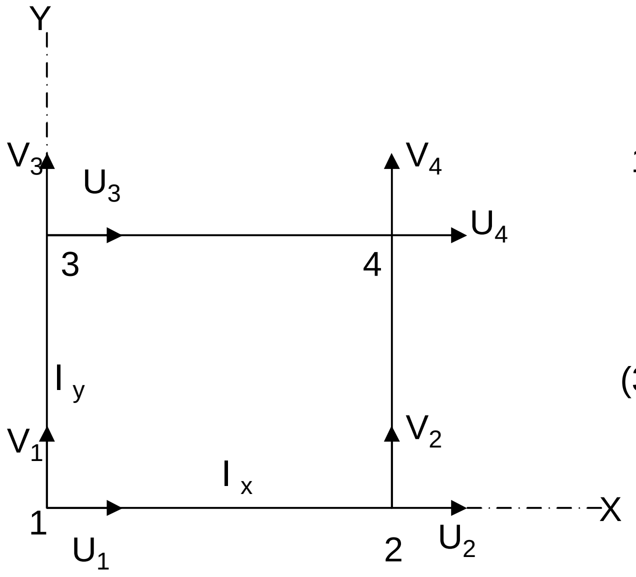

Fig.2.7 Rectangular plate for in-plane deformation

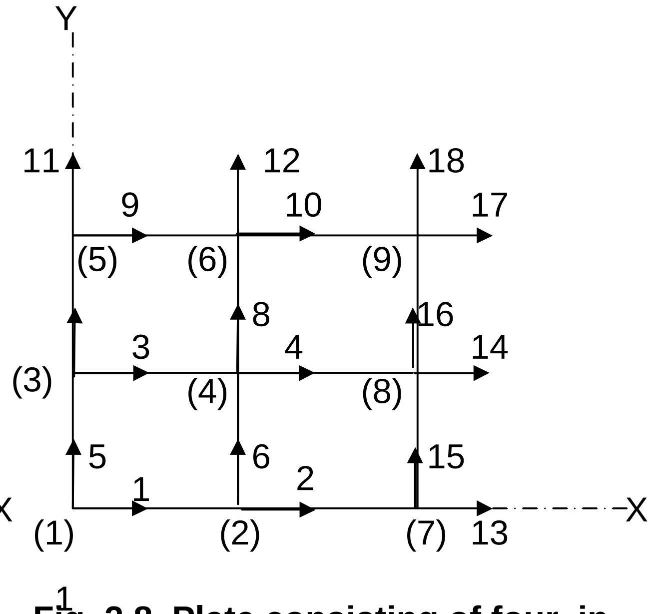

Fig. 2.8 Plate consisting of four in-plane deformation

the strain energy of the element

$$\delta_e = \frac{1}{2} h \int_0^{l_y}\int_0^{l_x} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_{xy} \varepsilon_{xy}) dxdy \tag{40}$$

the strain energy can be expressed in terms of displacements, using the stress-strain relations (12) and the strain-displacement relations (7) and (8),

$$\delta_e = \frac{1}{2}\frac{Eh}{1-v^2}\int_0^{l_y}\int_0^{l_x} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2v\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-v)}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right] dxdy$$

$$= \frac{1}{2}\int_0^{l_y}\int_0^{l_x} \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right] D \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y} \end{bmatrix} dxdy \tag{41}$$

The assumed displacement functions are

$$\left.\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 xy \\ v &= a_5 + a_6 x + a_7 y + a_8 xy \end{aligned}\right\} \qquad (42)$$

or

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix} a \qquad \text{where} \qquad g = \begin{bmatrix} 1 & x & y & xy \end{bmatrix}$$

and a is a vector containing the coefficients

$$a_1, a_2, \ldots\ldots a_8$$

$$u_e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = Na = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & l_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & l_y & 0 & 0 & 0 & 0 & 0 \\ 1 & l_x & l_y & l_x l_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & l_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & l_y & 0 \\ 0 & 0 & 0 & 0 & 1 & l_x & l_y & l_x l_y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \tag{43}$$

Thus $a = Bu_e$

where $B = N^{-1}$

As

$$\partial v / \partial y = a_7 + a_8 y$$

$$\partial u / \partial x = a_2 + a_4 y,$$

and

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = a_3 + a_4 x + a_6 + a_8 y$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \end{bmatrix} = Ga$$

where

$$G = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \qquad (44)$$

Substituting from equation (44) in equation (41)

$$\mathrm{d}_e = \frac{1}{2}\int_0^{l_y}\int_0^{l_x} a^T G^T D G a\,dxdy \;= \frac{1}{2} u^T{}_e B^T \left[\int_0^{l_y}\int_0^{l_x} G^T D G\,dxdy\right] B u_e$$

$$= \frac{1}{2} u^T{}_e K_e u_e$$

where the element stiffness matrix

$$K_e = B^T \left[\int_0^{l_y}\int_0^{l_x} G^T D G\,dxdy\right] B \qquad (45)$$

The kinetic energy of the element

$$\Im_e = \frac{1}{2} \mathrm{r}\, h \int_0^{l_y} \int_0^{l_x} \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] dy dx$$

$$= \frac{1}{2} \int_0^{l_y} \int_0^{l_x} \left[\frac{\partial u}{\partial t} \; \frac{\partial v}{\partial t} \right] \mathrm{r}\, h \begin{bmatrix} \dfrac{\partial u}{\partial t} \\ \dfrac{\partial v}{\partial t} \end{bmatrix} dy dx$$

$$= \frac{1}{2} \dot{u}_e^T M_e \dot{u}_e \qquad (46)$$

where the element mass matrix

$$M_e = B^T \mathrm{r}\, h \int_0^{l_y} \int_0^{l_x} \begin{bmatrix} g^T g & 0 \\ 0 & g^T g \end{bmatrix} dy dx B \qquad (47)$$

3.4 Finite Element Method for Transverse Vibrations of Plates

- The assumed displacement function is

$$w = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} xy^3 = ga$$

Eqt.(48)

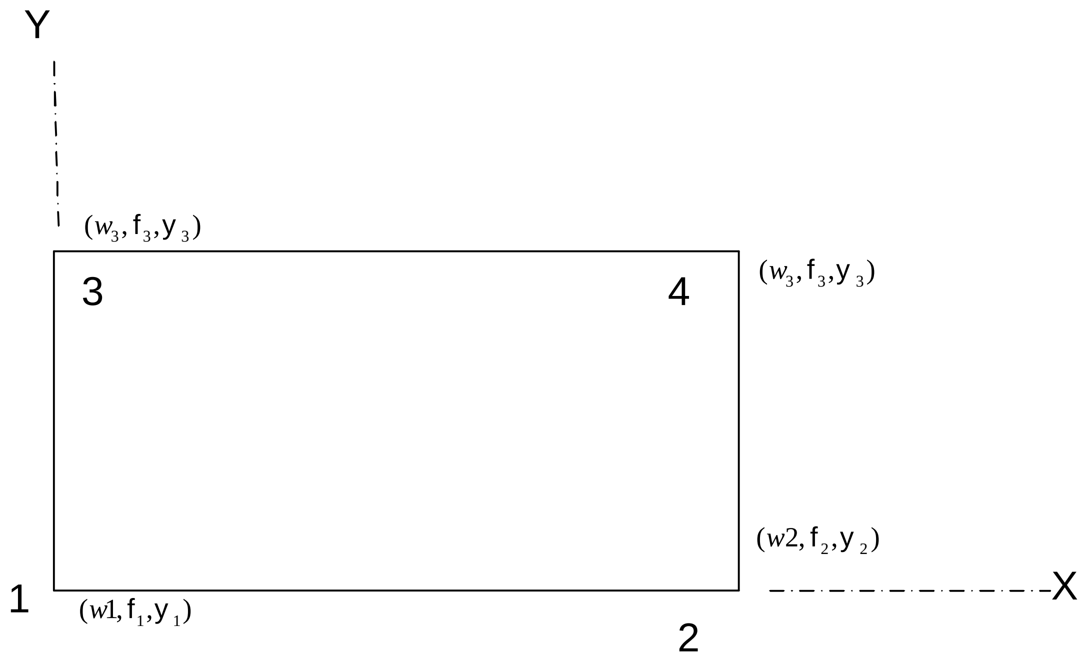

Fig.2.9 Rectangular plate for flexure of a plate

Substituting nodal values of x and *y* in equation (48),

$$w_e = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ \mathrm{f}_1 \\ \mathrm{f}_2 \\ . \\ . \\ . \\ \mathrm{y}_1 \\ \mathrm{y}_2 \\ . \\ . \\ . \end{bmatrix} = Na$$

$$a = N^{-1} w_e = B w_e \qquad (49)$$

From equation (13) the strain energy expression for the element can be written

$$\mathrm{d}_e = \frac{1}{2}\int_0^{l_y}\int_0^{l_x} \left[\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \left(\frac{2\partial^2 w}{\partial x \partial y}\right)\right] C \begin{bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{2\partial^2 w}{\partial x \partial y} \end{bmatrix} dx dy \qquad (50)$$

where

$$C = D\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-v) \end{bmatrix}$$

From Equation (48)

$$\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{2\partial^2 w}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix} a = Ga \qquad (49)$$

Using Eqs. (49) and (51) in equation (50)

$$\mathrm{d}_e = \frac{1}{2} w^T{}_e K_e w_e$$

from Eqs. (48 where the stiffness matrix

$$K_e = B^T \left[\int_0^{l_y} \int_0^{l_x} G^T C G \, dx dy \right] B$$

The kinetic energy of the element

$$\Im_e = \frac{1}{2} \int_0^{l_y} \int_0^{l_x} \rho h \left(\frac{\partial w}{\partial t} \right)^2 dy dx$$

(48) and (49)

$$\frac{\partial w}{\partial t} = g B \dot{w}_e$$

Thus

$$\Im_e = \frac{1}{2} \dot{w}_e^T M_e \dot{w}_e$$

where the element mass matrix

$$M_e = B^T \left[\rho h \int_0^{l_y} \int_0^{l_x} g^T dy dx \right] B \qquad (53)$$

For forced vibration the matrix equation

$$M\ddot{w} + Kw = p \qquad (54)$$

Considering the con-tribution to **p** from a particular element, $\mathbf{p}_e$, and supposing that this element is subjected to a transverse applied force per unit are a of

$$p(x, y)\, f(t)$$

,application of the principle of virtual work gives.

$$p^T{}_e \mathrm{d} w_e = \int_0 \int_0 p(x, y) f(t) \mathrm{d} w(x, y) dy dx$$

$\mathrm{d}\, w_e$ list the virtual increments in the element nodal values and

$\mathrm{d}\, w(x, y)$ is the virtual displacement ay point (x,y). Using the Eqs. (48) and (49),

$$p^T{}_e \mathrm{d}\, w_e = f(t) \int_0^{l_y} \int_0^{l_x} p(x, y) g B \mathrm{d}\, w_e \, dy dx$$

and

$$p^T{}_e = f(t) \left[\int_0^{l_y} \int_0^{l_x} p(x, y) g \, dy dx \right] B \qquad (55)$$

PART IV.

- References
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