Course “Trees – The Ubiquitous Structure in Computer Science and Mathematics”, JASS’08

The Number of Spanning Trees in a Graph

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April 28, 2008

Definition 1

Let $G = (V, E)$ with $V = \{1, \dots, n\}$ and $E = \{e_1, \dots, e_m\}$ be a directed graph. Then the incidence matrix $S_G \in M(n, m)$ of G is defined as:

$$(S_G)_{i,j} := \begin{cases} 1 & \text{if } e_j \text{ ends in } i \\ -1 & \text{if } e_j \text{ starts in } i \\ 0 & \text{else} \end{cases}$$

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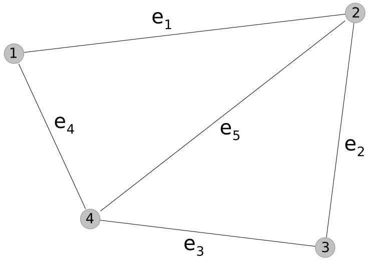
Remark 1

For an undirected graph G every $S_{\bar{G}}$ of some arbitrarily oriented directed variant $\bar{G}$ of G can be taken as the incidence matrix.

Example 2



$$S_G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

Theorem 3

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Proof.

Reorder the edges and vertices so that:

$$S_G = \begin{pmatrix} S_{G_1} & & \dots & 0 \\ & S_{G_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & S_{G_r} \end{pmatrix}$$

$\square$

Remark 2

Since $(1, \ldots, 1) \cdot S_G = 0$, we can remove an arbitrary row from S_G without losing information.

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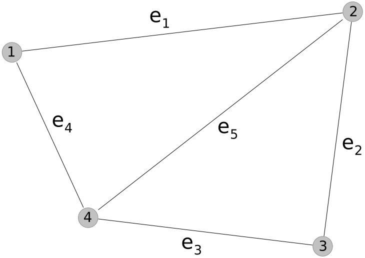
Definition 4

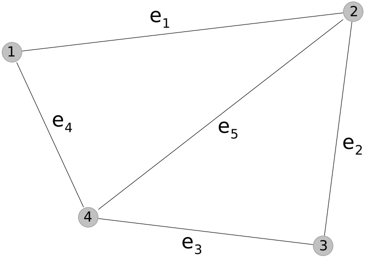
For every $A \in M(n, m)$ define $\tilde{A} \in M(n-1, m)$ as A without the n-th row.

Example 5



$$\tilde{S}_G \cdot \tilde{S}_G^T =$$

Example 5



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$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Theorem 6 (Kirchhoff)

The number of spanning trees of a graph G can be calculated as:

$$\det(D_G) \text{ where } D_G = \tilde{S}_G \cdot \tilde{S}_G^T$$

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Remark 3

$$(D_G)_{i,j} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if } \{i,j\} \in E \\ 0 & \text{else} \end{cases}$$

Lemma 7

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Remark 4

Every undirected tree on V has exactly one undirected variant that is rooted at n. So for constructing/counting spanning trees we only have to consider graphs with $i \in e_i$.

Lemma 8

Let $G = (V, E)$ *with* $|E| = n - 1$ *be a directed graph which is not a tree.*
Then $\det(\tilde{S}_G) = 0$.

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Proof.

Since $|E| = n - 1$, G is not "weakly" connected.
So $\operatorname{rank}(\tilde{S}_G) = \operatorname{rank}(S_G) \leq n - 2$. $\square$

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Proof.

Order the vertices (and edges simultaneously – e_i has to end in i) so that $p(i) > i$. Then,

$$\tilde{S}_T = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

□

Proof of Kirchhoff's theorem.

We observe that the i-th column of D_G is the sum of "incidence vectors" that correspond to edges in G that have endpoints in the i-th vertex.

Figure: First column of D_G

$$D_{.,1} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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We observe that the i-th column of D_G is the sum of "incidence vectors" that correspond to edges in G that have endpoints in the i-th vertex. If we now use the linearity of the determinant in every column we obtain:

Figure: Expansion of the determinant

$$\det(D) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & -1 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ 0 & 4 & -1 \\ -1 & -1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \dots$$

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$$\det(D_G) = \sum_{H \in \mathcal{H}} \det(\tilde{S}_H)$$

where $\mathcal{H}$ is the set of all subgraphs of G which correspond to a selection of $n-1$ edges, with e_i ending in i.

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where $\mathcal{H}$ is the set of all subgraphs of G which correspond to a selection of $n-1$ edges, with e_i ending in i.
To prove the theorem it is now sufficient to use the preceding lemmata. $\square$

Definition 10

For a weighted graph G with edge weights $w_{i,k}$ let $S_G(x)$ be the incidence matrix S_G with every column corresponding to $(i,k) \in E$ rescaled by $x^{w_{i,k}}$.

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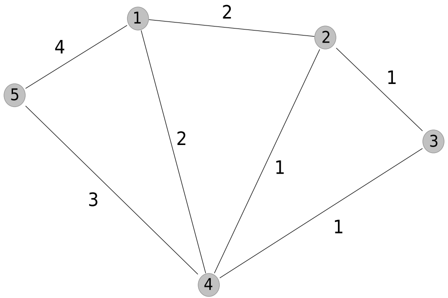
$$(S_G(x))_{i,j} = \begin{cases} x^{w_{k,i}} & \text{if } e_j = (k,i) \\ -x^{w_{i,k}} & \text{if } e_j = (i,k) \\ 0 & \text{else} \end{cases}$$

Example 11 (Toy graph H)



$$S_G(x) = \begin{pmatrix} -x^2 & -x^2 & -x^4 & 0 & 0 & 0 & 0 \\ x^2 & 0 & 0 & -x & -x & 0 & 0 \\ 0 & 0 & 0 & x & 0 & -x & 0 \\ 0 & x^2 & 0 & 0 & x & x & -x^3 \\ 0 & 0 & x^4 & 0 & 0 & 0 & x^3 \end{pmatrix}$$

Theorem 12 (Matrix-Tree Theorem for weighted graphs)

The generating function of the number of spanning trees by weight w is the determinant of

$$D_G(x) = \tilde{S}_G(x) \cdot \tilde{S}_G^T$$

where $\tilde{S}_G(x)$ and $\tilde{S}_G$ are defined as above.

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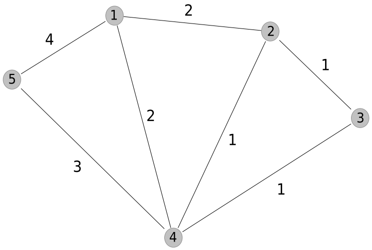
where $\tilde{S}_G(x)$ and $\tilde{S}_G$ are defined as above.
In other words:

$$\det\left(D_G(x)\right) = \sum_{w=0}^{\infty} |ST \text{ by weight } w| \cdot x^w$$

Example 13



$$D_G(x) = \begin{pmatrix} x^4 + 2x^2 & -x^2 & 0 & -x^2 \\ -x^2 & x^2 + 2x & -x & -x \\ 0 & -x & 2x & -x \\ -x^2 & -x & -x & x^3 + x^2 + 2x \end{pmatrix}$$

$$\det\left(D_G(x)\right) = 2x^{10} + 5x^9 + 8x^8 + 6x^7$$

Remark 5

It is easy to check that we can also write down $D_G(x)$ for any given graph G directly:

$$(D_G(x))_{i,j} = \begin{cases} \sum_{\{i,k\} \in E} x^{w_{i,k}} & \text{if } i = j \\ -x^{w_{i,j}} & \text{if } \{i,j\} \in E \\ 0 & \text{else} \end{cases}$$

Proof.

As above.

Here each spanning tree by weight w contributes x^w to the determinant of $D_G(x)$.

$\square$

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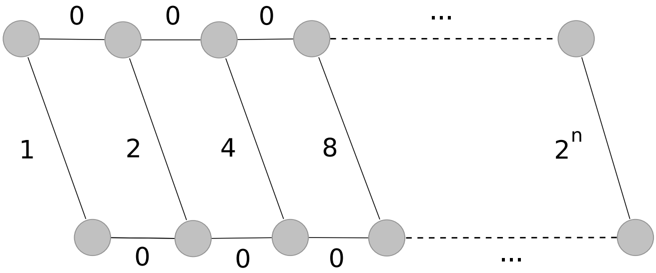
- Consider the following graph on $2n$ vertices:



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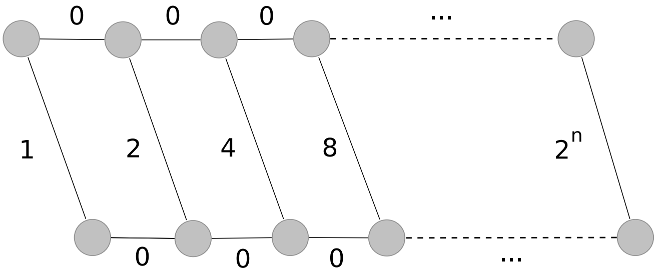

- $\det(D_G(x))$ can have $\Omega(2^n)$ coefficients

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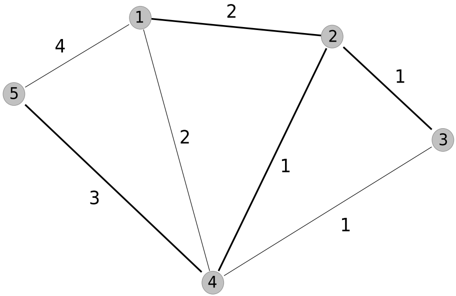
- If $w_{\min}$ is the minimal weight for a spanning tree, clearly $x^{w_{\min}}$ must divide $\det(D_G(x))$
- Try to factor out the minimum degree monomial of each column and use the linearity of the determinant.

$$det(D_G(x)) =$$

$$\begin{vmatrix} x^4+2x^2 & -x^2 & 0 & -x^2 \\ -x^2 & x^2+2x & -x & -x \\ 0 & -x & 2x & -x \\ -x^2 & -x & -x & x^3+x^2+2x \end{vmatrix} = x^5 \cdot \begin{vmatrix} x^2+2 & -x & 0 & -x \\ -1 & x+2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & x^2+x+2 \end{vmatrix}$$

Example 14 (Factor $\det(D_G(x))$)



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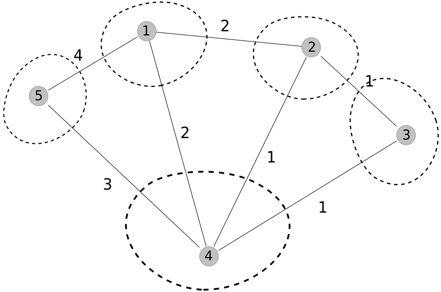
The entries of the i'th column of $D_G(x)$ correspond to edges in the cut $(i, \mathcal{N}(i))$:



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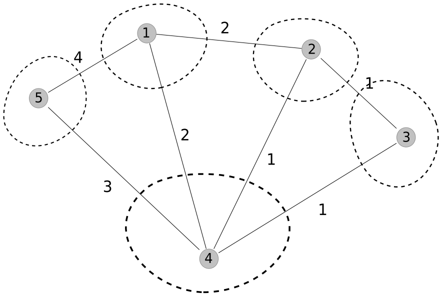

Change the cuts?

Theorem 15

If we have a minimal spanning tree T of G, we can modify $D_G(x)$ (without changing the determinant) so that the product of the minimum degree monomials of each column is w_{min}.

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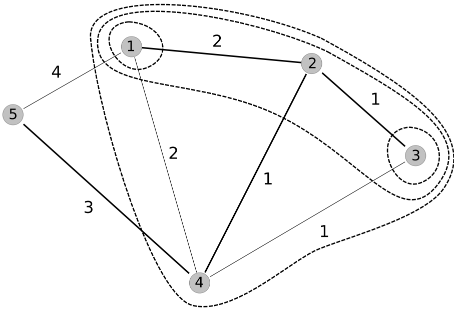
Algorithm A: Modify $D_G(x)$

T := mst(G)
$D'(x)$:= $D_G(x)$
while $T \neq \{\}$ do
    i := arbitrary leaf of T with $i \neq n$
    p := parent of i
    add the i-th column in D' to the p-th column
    T := $T \setminus \{i\}$
od

Example 16 (MST of H and corresponding $\sigma(i)$)



$\det(D'(x)) =$

$$\begin{vmatrix} x^4+2x^2 & x^4+x^2 & 0 & x^4 \\ -x^2 & x & -x & 0 \\ 0 & x & 2x & 0 \\ -x^2 & -x^2-2x & -x & x^3 \end{vmatrix} = x^7 \cdot \begin{vmatrix} x^2+2 & x^3+x & 0 & x \\ -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & -x-2 & -1 & 1 \end{vmatrix}$$

Lemma 17

The i-th column of $D'(x)$ contains the sum of the columns in $D_G(x)$ corresponding to $\sigma(i)$.

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$$D'_{i,j}(x) = -\sum_{\{i,k\}\in E : k \in \sigma(j)} x^{w_{k,i}}$$

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For $i \in \sigma(j)$ the above cancels with $D_{i,i}$

$$D'_{i,j}(x) = \sum_{\{i,k\} \in E : k \notin \sigma(j)} x^{w_{k,i}}$$

Proof.

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The only edge of T in the j'th cut is $(p(j), j)$.
All the other edges from the cut must have higher weight because T is minimal.

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$$\prod_{i=1}^{n-1} w_{p(i),i} = w_{\min}$$

□

The runtime of our implementation so far is $O\left(mn + M(n)\right)$ where $M(n)$ is the time required to multiply two $n \times n$matrices.
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- only calculating the minimum degree monomials for each entry.
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In the following E is sorted so that $p(j) > j$.

For $i \notin \sigma(j)$ no entries cancel – the naive approach works.

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Algorithm B1: Negative Entries of D'

for $j = 1$ to $n - 1$ do
    for $i \notin \sigma(j)$ do
        $D'_{i,j}$:= min_d$(D_G(x)_{i,j} + \sum_{k \texttt{ child of } j \texttt{ in } T} D'_{i,k})$
        / min_d computes the minimum degree monomial */*
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This is $O(n^2)$.

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Algorithm B2: Positive Entries of D'

for $i = 1$ to $n-1$ do
    L := sort($\mathcal{N}(i)$)
    for $j = 1$ to $n-1$, $i \in \sigma(j)$ do
        L := $L \setminus \sigma(j)$
        if $L \neq \{\}$
            k := first_element(L)
            $D'_{i,j}$:= $x^{w_{i,k}} \cdot |s \in L : w_{i,k} = w_{i,s}|$
        fi
    od
od

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Algorithm B2: Positive Entries of D'

```
for i = 1 to n − 1 do
    L := sort(𝒩(i))
    for j = 1 to n − 1, i ∈ σ(j) do
        L := L \ σ(j)
        if L ≠ {}
            k := first_element(L)
            D'_{i,j} := x^{w_{i,k}} · |s ∈ L : w_{i,k} = w_{i,s}|
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```

This is $O(n^2 \log n)$.

Conclusion:

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- We can find the number of minimal spanning trees in $O(n^2 + m \log n + M(n)) = O(M(n))$