Tree reconstruction

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Abstract

The problem of graph reconstruction is open problem nevertheless it is so easy in formulation. But this problem is completly solved in the case of trees. This paper gives an overview of the proof.

Contents

1 Setting

For the needs of this paper all graphs will be simple and undirected. We won't differ the nodes and verticies. We will use representation of a graph as a pair of sets of its nodes (V) and edges (E). Such graph $G = (V_G, E_G)$

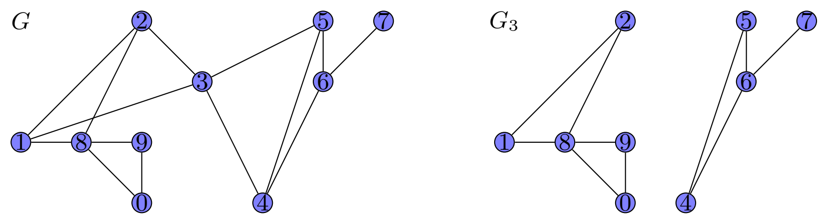
Definition. *$A \cong B$ if there is some bijection between V_A and V_B which preserves edges.*

Trees are connected graphs without cycles as usual. For every tree $|E| = |V| - 1$. Every two nodes can be connected in a single way. This properties of any tree can be easily estimated by induction.

2 Main theorem

Main issue uses in it's formulation parts of the graphs received from the original graph by deletion of one vertex. We will denote it in following way.

Definition. *Let G be a graph consisting of (V, E). Let $V_v = V \setminus \{v\}$*
Then with G_v we denote $(V_v, E|_{V_v \times V_v})$



One question arises in connection with that procedure: how many information we lose after removal of one vertex? Of course we can't estimate anything about removed vertex in case that we have only one subgraph. But what happens if we have more then one? Following definiton also will be useful.

Definition. *Deck D_G of graph G is a multiset $\{G_v | v \in V_G\}$*

Main conjecture that it is sufficient information in deck to recover the graph completely.

Conjecture. *(Reconstruction conjecture) Let graphs A and B have $n > 2$ vertices and $D_A = D_B$. Then $A \cong B$.*

Nevertheless it's so simple in formulation, it's still an open question. But if add some restriction, we come to the complete result.

Theorem 1. *Let trees A and B have $n > 2$ vertices and $D_A = D_B$. Then $A \cong B$.*

3 Proof

First of all we recall some famous notion and properies connected to trees.

Definition. *Diameter is the path of maximal length.*

Of course, in the tree can be more then one diameter, as in the star-tree, where only one vertex isn't pendant. With d we will denote the length of diameter.

Definition. *Centers are the points with minimal maximal distance to the pendant vertices.*

From this definition clear that there is at most two centers in any tree. This is because every center should lie on this maximal distance path from any other one.

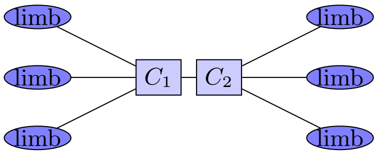
Definition. *Radius r of tree is the maximal distance from the pendant vertex to the closest center.*

It's easy to see that $r = \left\lfloor \frac{d}{2} \right\rfloor$ We should just pay attention to the middle points in any diameter. This bring us to the next

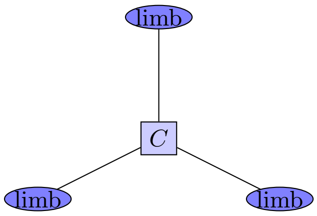
Proposition. *If tree has odd diameter then there are two centers, else there is one center.*

With respect to this proposition we can obtain an useful information about tree structure.

Odd diameter trees has two centers. Every diameter goes through them.



Even diameter has one center. Every diameter can be splited into two parts of semilength in different limbs.



The remaining part are closer to the proof of main theorem. In it we denote with A and B two graphs from the statement of main Theorem **??**. So they has the same decks, and let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be an agreed numerations: $A_{a_i} \cong B_{b_i}$.

Lemma 1. *Let A be a graph on n vertices, T is a graph on j vertices. Let α_i is the number of T-type subgraphs, containing a_i. Then for the total number α of T-type subgraphs the following holds:*

$$\alpha = \frac{\sum_{i=1}^n \alpha_i}{j}$$

Proof. For the proof we calculate the total number N of verticies in all T-type subgraphs in two ways. From the first side, counting the entries of each vertex separately, we recive $N = \sum_{i=1}^n \alpha_i$. From the second side N simply equals $\alpha|T| = \alpha j$. From here we recive the claim of lemma. □

Corollary. *Let A and B be a graphs on n vertices and T be a graph on $j < n$ vertices, then $\alpha_{a_i} = \beta_{b_i}$*

Proof. Since $A_{a_i} \cong B_{b_i}$, T occurs in each the same number of times: $\alpha - \alpha_{a_i} = \beta - \beta_{a_i}$

Sum it up: $\sum_{i=1}^n (\alpha - \beta) = \sum_{i=1}^n (\alpha_{a_i} - \beta_{a_i})$

After applying the lemma **??** we receive: $n(\alpha - \beta) = j(\alpha - \beta)$, and then $\alpha = \beta$. After removing of it from the first equality for all i we receive $\alpha_{a_i} = \beta_{b_i}$. □

Corollary. *Let A and B be a graphs on n vertices. Then* $\deg(a_i) = \deg(b_i)$.

Proof. It follows from previous corollary for T consisting of one edge. □

Lemma 2. *A and B are of the same diameter. So they have the same radius and central or bicentral simultaneously.*

Proof. Case 1. A and B are paths. It's sufficient to delete extreme vertex in this case.

Case 2. In each graph exists vertex of degree at least 3. Assume, for example, then in A diameter is bigger. Then we can remove vertex in A such that at least one diameter preserves. This means that in B exists path of at least the same length. After the symmetrical reasoning we receive that diameters are the same. □

We will classify different cases according to the pieces of graphs wich are hangs on the centers.

Definition. *Let c be a center of A. Let F' be a component of A_c which is connected with c in A. Assume $F = F' \cup \{c\}$ Then $(F, E|_{F\times F}))$ is the limb at c.*

Significant property of limb is presence of remote point in it.

Definition. *Let r be a radius of tree. r-point is the point which have distance r to the closest center.*

Definition. *Limb is radial if it contains an r-point. Otherway, it is nonradial.*

Lemma 3. *a_i is r-point if and only if b_i r-point.*

Proof. If graphs are $2r + 1$ paths it is trivial. Otherwise it's the colloary of lemma **??** for T equals $2r + 1$ path. □

In both considered in the proof cases we can see, that the number of diameters going through coressponding vertices are the same.

We give a sketch of a proof for harder case of bicentral tree. Case of one center tree can be investigated in a similar maner. Let the centers of A will be $\bar{a_1}$ and $\bar{a_2}$, and the centers in B will be $\bar{b_1}$ and $\bar{b_2}$.

For later needs we split A into parts: A_i - all radial limbs at $\bar{a_i}$, B_i - all nonradial limbs at $\bar{a_i}$.

The same thing with B: C_i - all radial limbs at $\bar{b_i}$ and D_i - all nonradial at $\bar{b_i}$.

Also we denote $A_r = A_1 \cup A_2$, $B_r = B_1 \cup B_2$, $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$.

Definition. *Pendant point $a \in A$ called nonesential(n.e.) if in A_a at least one diameter preserves.*

It implies that A_a also bicentral with the same radius.

For example, all pendant non r-points are n.e. points. Also, all points which are belonged to a center with at least 2 radial limbs are n.e. points too. And all r-points which a not single r-points in limb are n.e. points.

From corollary of lemma **??** follows that a_i and b_i are of the same type.

The followong proofs isn't complete. They consists of just main ideas.

More ever,

Lemma 4. *If a_i is n. e. in A_r then b_i is n. e. in B_r.*

Proof. If we assume contoroary, then we first come to the following: $|A_r| = |B_r| + 1$ and $A = A_r$, and sequentially $|D| = 1$. After the series of observation we come to the fact, that A_r differs from $2r + 1$ path in one vertex □

Proof. (of main theorem)

The proof essentialy splits in 3 cases. Main game comes around radial parts of graphs.

Case 1: $C \neq \emptyset$

If we remove any point from nonradial limb, we recive that all radial limbs should go to the radial. Repeating this reasoning for both graphs, we receive $A_r \cong B_r$. After the series of observation we come to the fact, that there is a center preserving congruence of C and D.

If this congruences are consistent, then we are done. Otherwise we should pay attention to nonradial parts.

Case 2: $C = \emptyset$, A_r consist of at least 3 limbs

In this case main idea consist of deletion of one of the centers. After that we recives isomorphism of parts of the graphs wich hangs on the centers of degree at least 2, say $A_2 \cong B_2$. If first two are also of degree at least 2, then it's enough just to repeat this idea to another center. Otherwise it's enough to delete some r point in isomorphic parts.

Case 3: $C = \emptyset$, A_r consist of 2 limbs

First we receives that sizes of parts are bounded with following relation $|A_1| = |B_1|$ and $|A_2| = |B_2|$. Then we should consider n. e. points A_i and show thet corresponding shoud lie in B_i. After that we should pay attention to n. e. points in smallest part. □

References

[1] Paul J. Kelly. A congruence theorem for trees