2-D Reconstruction

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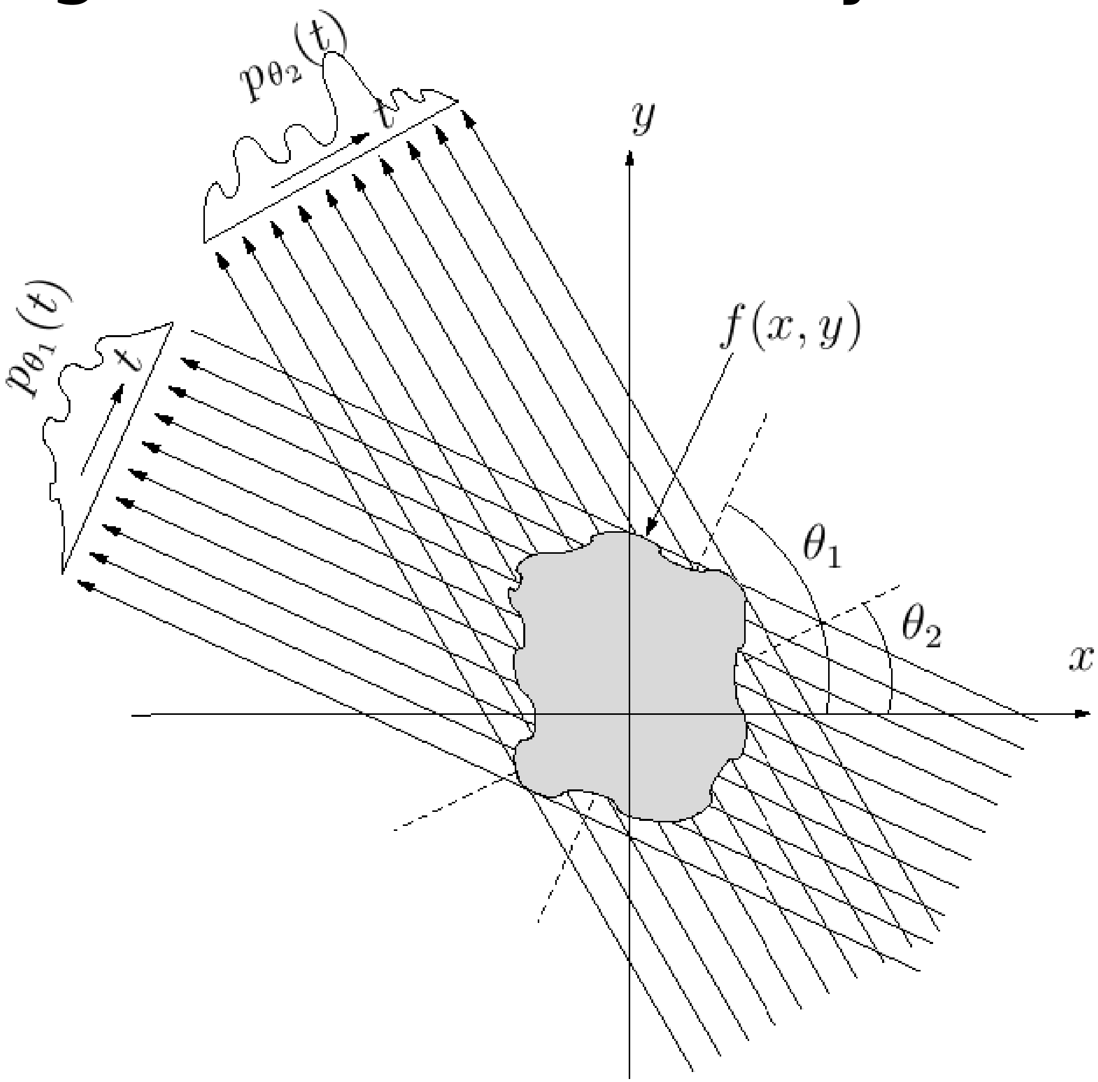
Outline

- Projections
- Radon Transform
- Fourier-Slice-Theorem
- Filtered Backprojection
- Ramp Filter

Parallel Projections

X-Rays are attenuated as they propagate through the human body



$p_\theta(t)$: projection under viewing angle θ

$f(x,y) = f(\boldsymbol{x})$: 2D slice

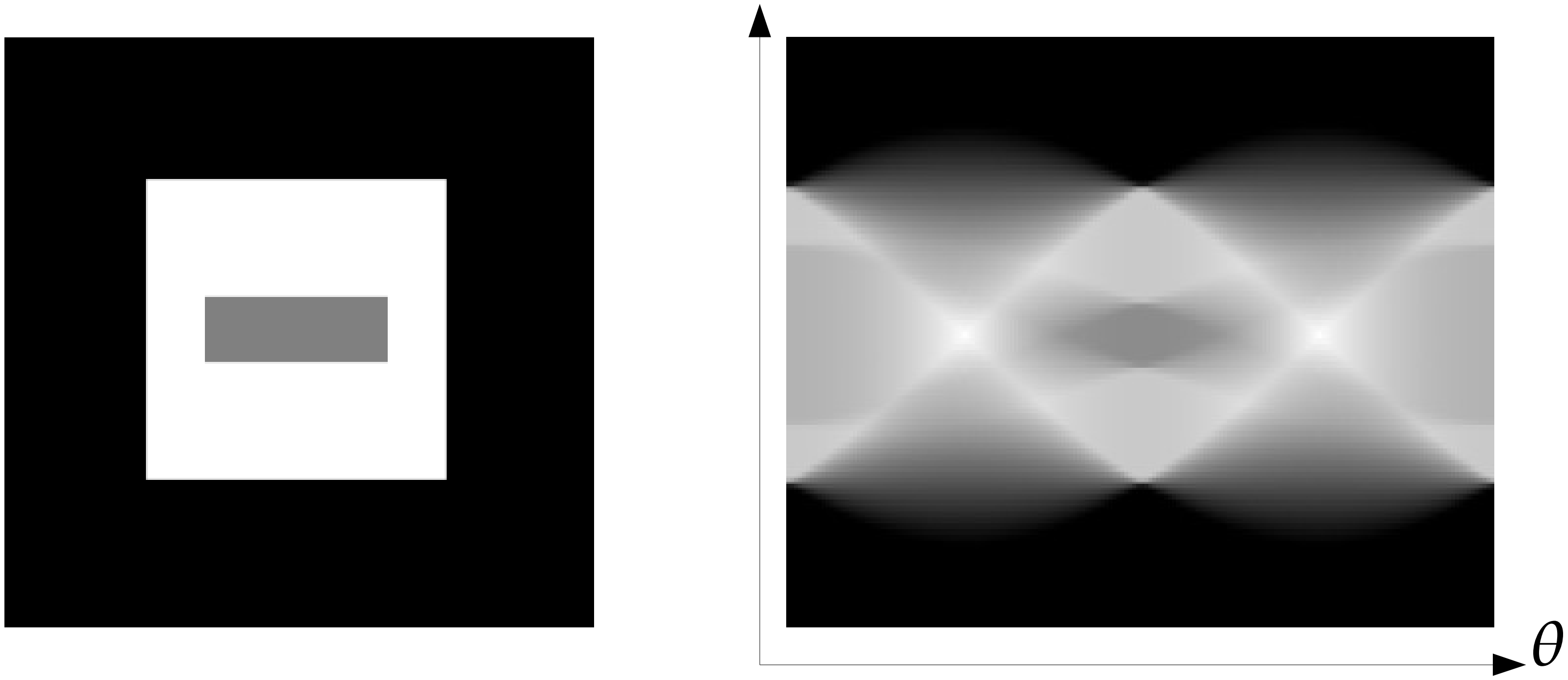
Radon Transform

- The signal $p_\theta(t)$ is the Radon transform of the object

$$p_\theta(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta+y\sin\theta-t)\,dx\,dy$$

- Set of line integrals along the direction θ at the distance t from the origin

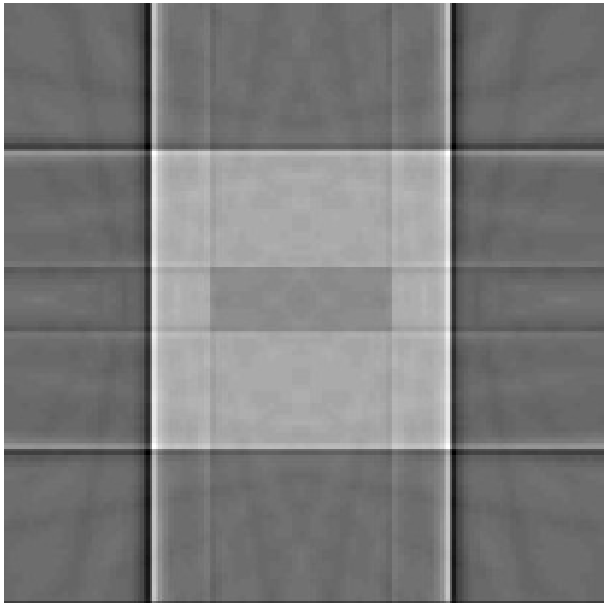
Radon Transform (2)



A simple image (left) and the sinogram (right)

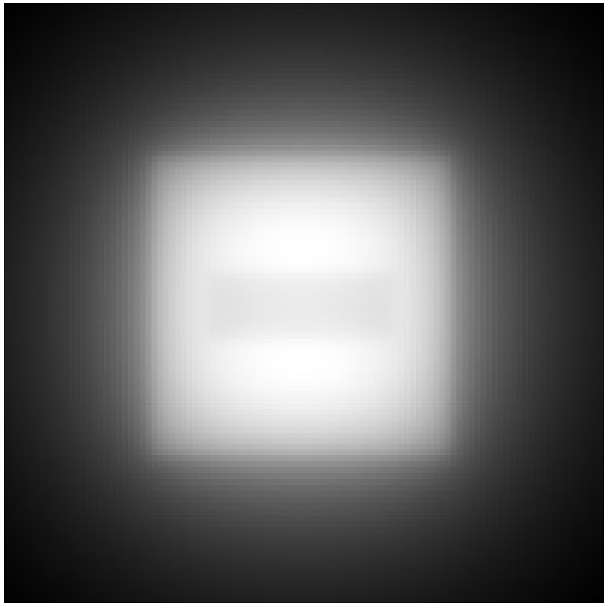
produced by applying the Radon transform
(180 projections with an angle of 1 degree)

Unfiltered BP



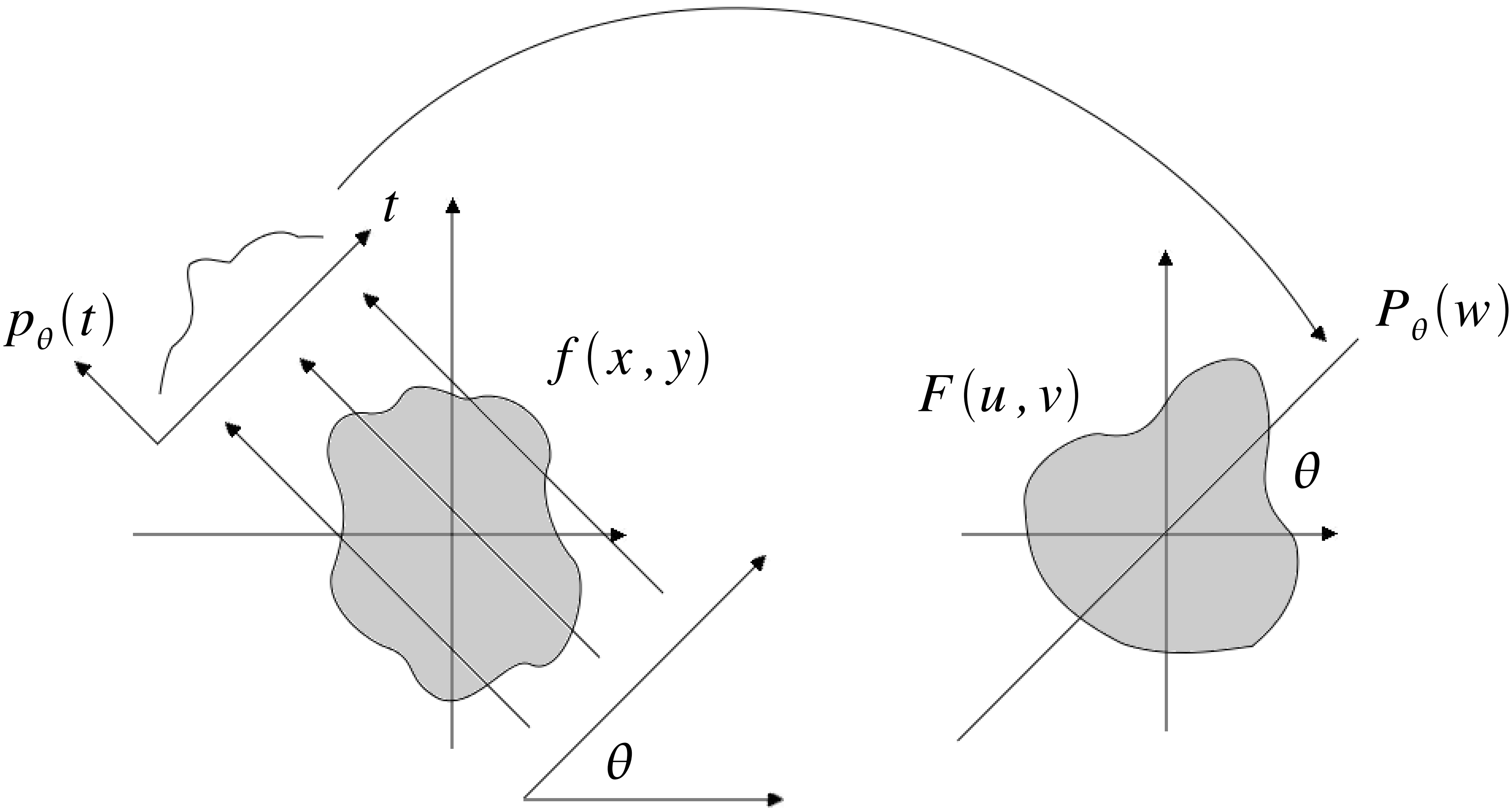
Ok, that's bad. But we used only 18 projections.

Unfiltered BP (2)



Now we used all 180 projections.
Still poor image quality.

Fourier Slice Theorem



Projection under angle θ

Slice under θ in freq. domain

Fourier Slice Theorem (2)

- Projection of $f(x, y)$ in y direction is $p_{\theta=0}(x)$

$$p_{\theta=0}(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

Fourier Slice Theorem (2)

- Projection of $f(x,y)$: $p_{\theta=0}(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$
- The Fourier transform of $f(x,y)$ is

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,e^{2\pi i(xu+yv)}\,dx\,dy$$

Fourier Slice Theorem (2)

- Projection of $f(x,y)$: $p_{\theta=0}(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$
- The Fourier transform of $f(x,y)$ is

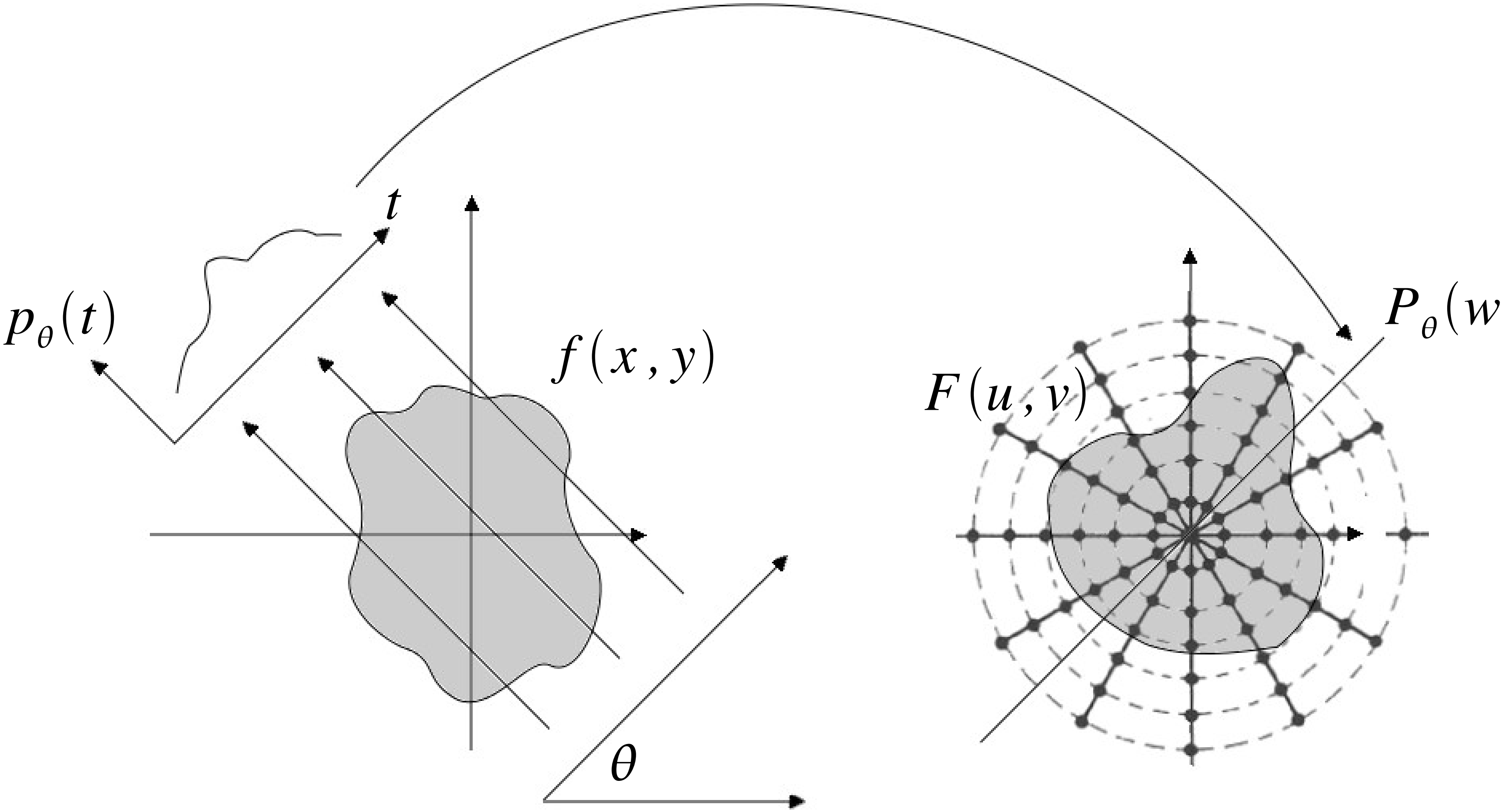
$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{2\pi i(xu+yv)}\,dx\,dy$$

- The slice $s(u)$ is then

$$s(u) = F(u,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi xu}\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right] e^{-i2\pi xu}\,dx$$
$$= \int_{-\infty}^{\infty} p_{\theta=0}(t)\, e^{-i2\pi tu}\,dt$$

- which is just the Fourier transform of $p_{\theta=0}(x)$
- Formal derivation see e.g. Kak & Slaney

Fourier Slice Theorem (3)



Projection under many angles

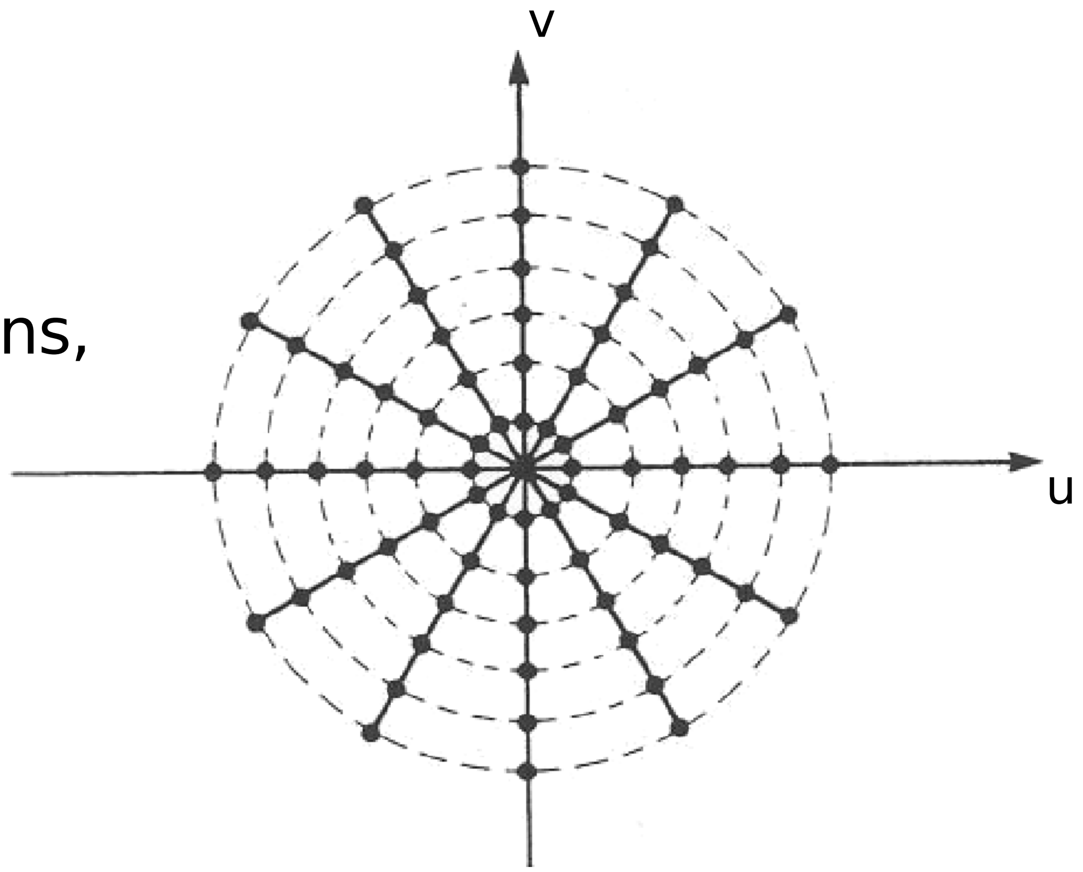
Slices in frequency domain

FST - Regridding

Projection data lies on circles (dots) and has to be interpolated to a Cartesian coordinate system

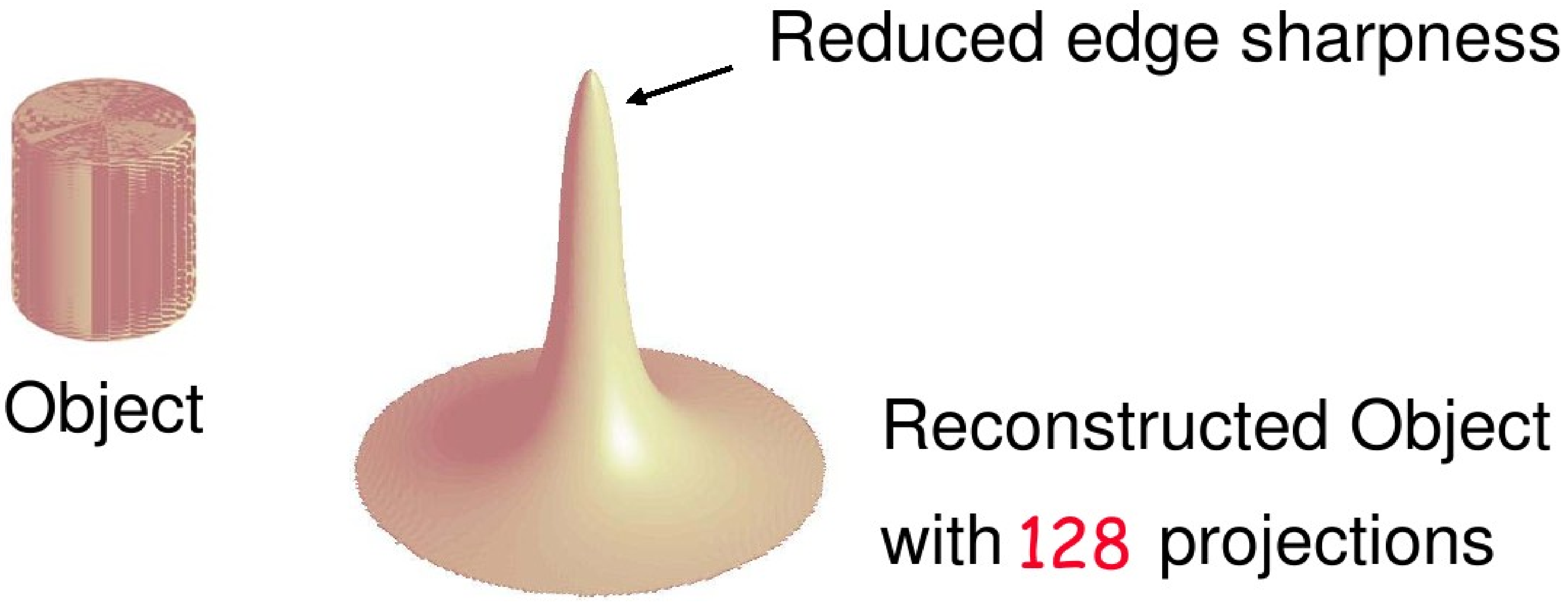
Interpolation gets worse with increasing u & v

Errors in high-frequency regions, i.e. regions with high detail

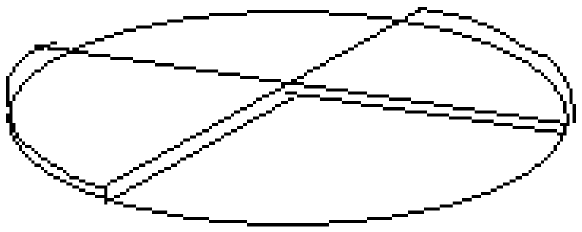


Frequency domain

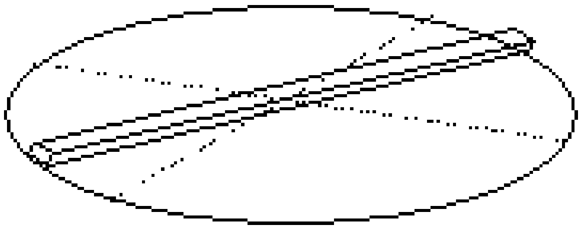
Unfiltered Backprojection



Solution: Filtering

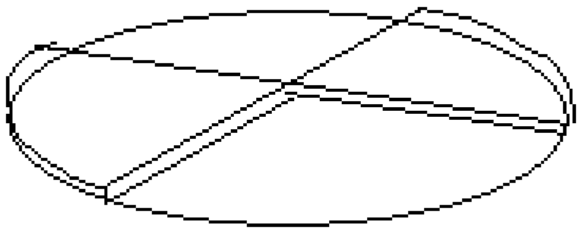


Ideal situation

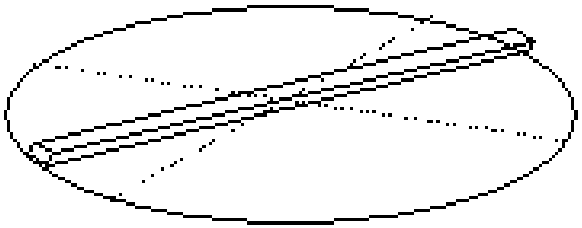


Fourier Slice Theorem

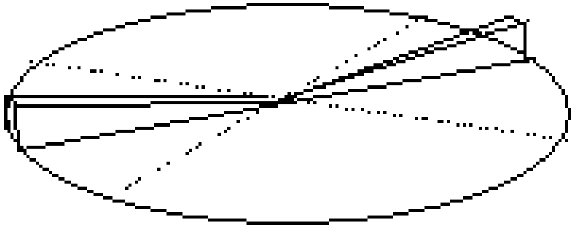
Solution: Filtering



Ideal situation



Fourier Slice Theorem



The ideal is approximated by a weighting (highpass)

Filtered Backprojection

- Switch to polar coordinates

$$f(x,y)=\int_0^{2\pi}\int_0^{\infty}F(w,\theta)e^{i2\pi(x\cos\theta+y\sin\theta)}w\,dw\,d\theta$$

$$=\int_0^{\pi}\int_0^{\infty}F(w,\theta)e^{i2\pi(x\cos\theta+y\sin\theta)}w\,dw\,d\theta$$

$$+\int_0^{\pi}\int_0^{\infty}F(w,\theta+\pi)e^{i2\pi(x\cos(\theta+\pi)+y\sin(\theta+\pi))}w\,dw\,d\theta$$

Filtered Backprojection

- Switch to polar coordinates

$$f(x,y)=\int_0^{2\pi}\int_0^{\infty} F(w,\theta)\, e^{i2\pi(x\cos\theta+y\sin\theta)}\, w\, dw\, d\theta$$

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$$+\int_0^{\pi}\int_0^{\infty} F(w,\theta+\pi)\, e^{i2\pi(x\cos(\theta+\pi)+y\sin(\theta+\pi))}\, w\, dw\, d\theta$$

- 180° symmetry

$$F(u,\theta+\pi)=F(-u,\theta)$$

$$f(x,y)=\int_0^{\pi}\left[\int_{-\infty}^{\infty} F(w,\theta)\, e^{i2\pi wt}\, |w|\, dw\right] d\theta$$

$$(t=x\cos\theta+y\sin\theta)$$

Filtered Backprojection (2)

- Apply Fourier Slice Theorem

$$f(x,y)=\int_0^{\pi}\left[\int_{-\infty}^{\infty}P_\theta(w)e^{i2\pi wt}|w|dw\right]d\theta$$

- Which can be written as

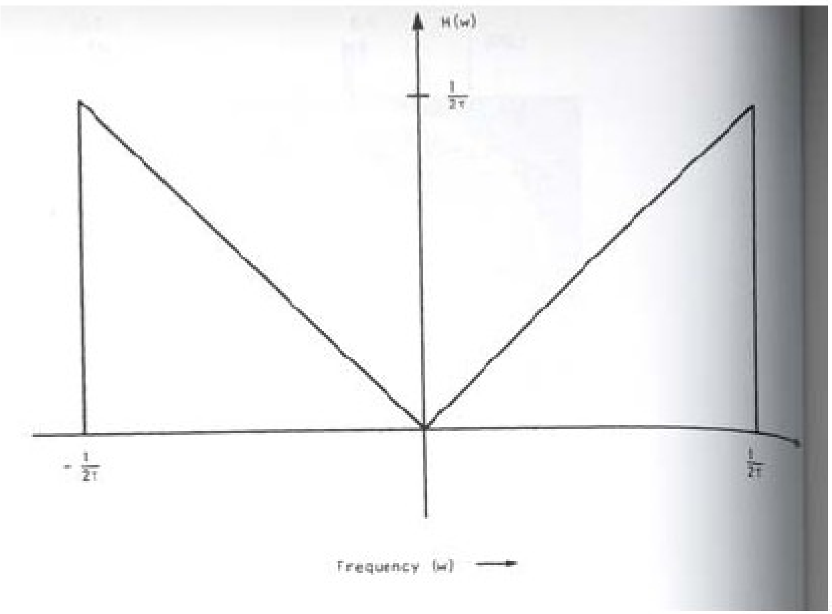
$$f(x,y)=\int_0^{\pi}Q_\theta(t)\,d\theta$$

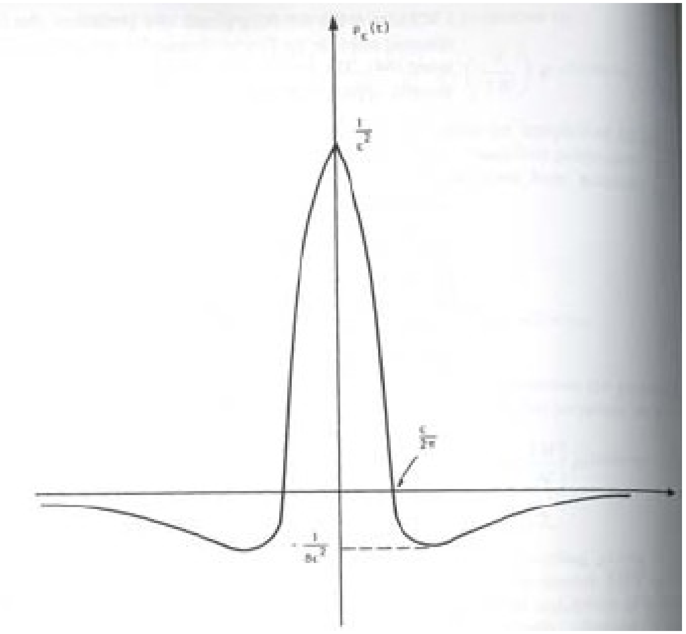
$$Q_\theta(t)=\int_{-\infty}^{\infty}P_\theta(w)e^{i2\pi wt}|w|dw$$

- Q is the filtered projection with filter function |w| (in frequency domain)

Ramp Filter

- Convolution in frequency domain
- High-pass, emphasizes noise

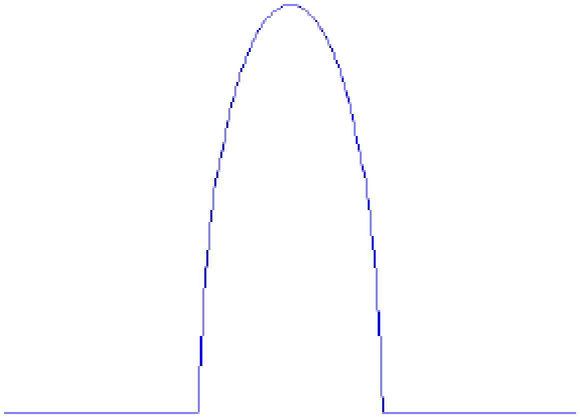




Filter Results

Attenuation profile of a cylinder

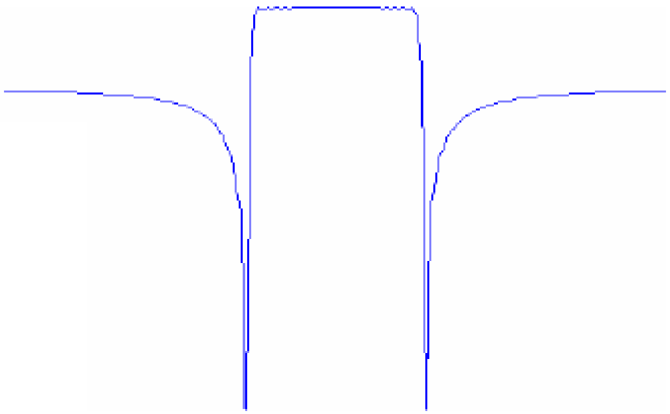
$$p_\theta(t)$$



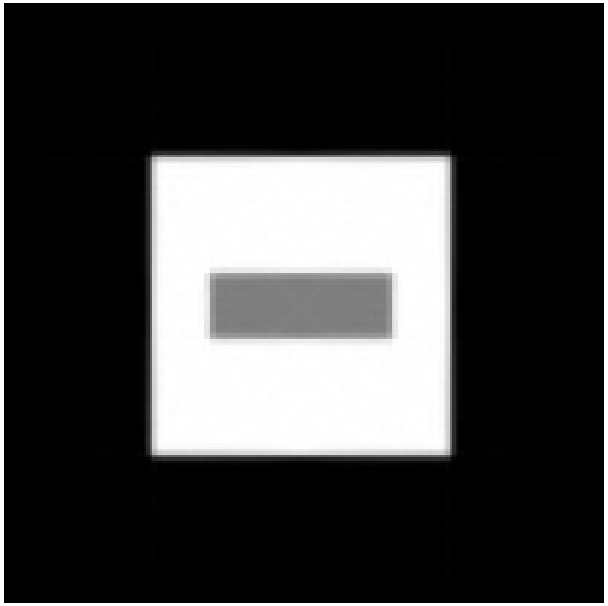
convolution

Filtered attenuation profile

$$p_\theta(t)\,K(t)$$



Filtered BP: Result



Reconstruction using
ramp filtered backprojection
(180 projections)

Sharp Kernel

Bone image:

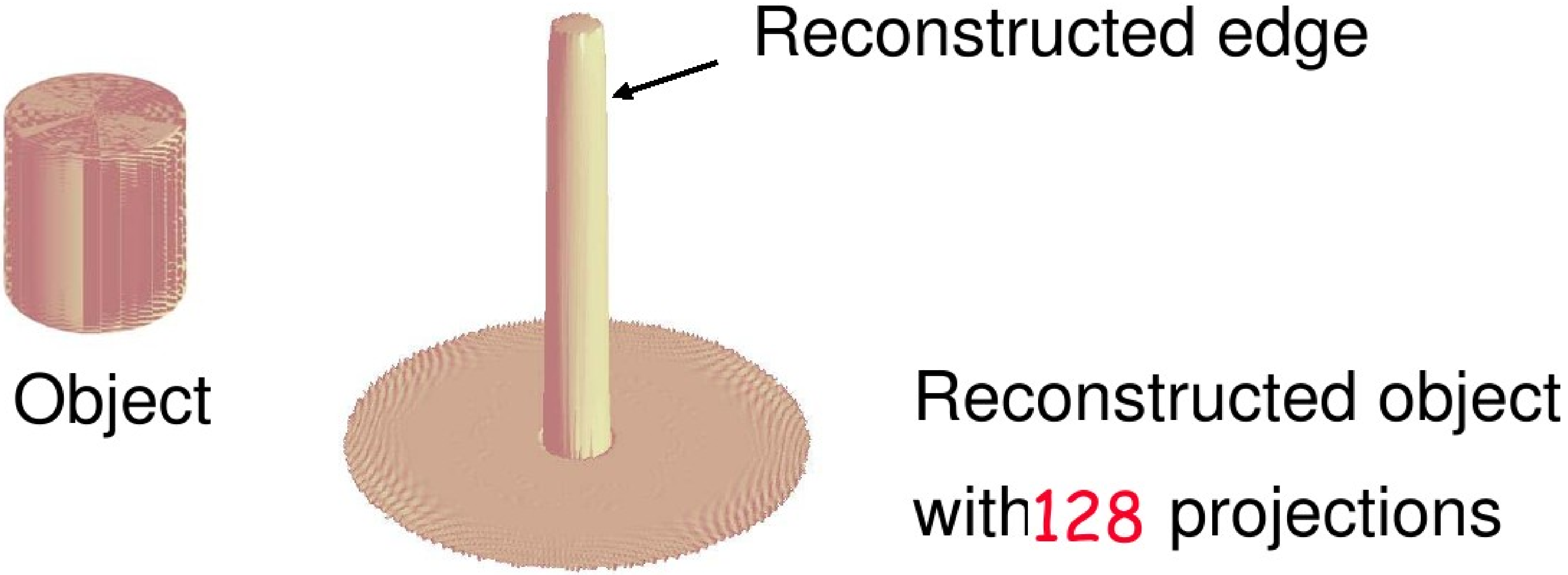
use sharp kernel
high resolution
high noise

Soft Kernel

Soft tissue:

use soft kernel
lower resolution
less noise

Filtered Backprojection



Thanks

Thank you for your attention.

Do you have any questions?

References

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