2D Fan Beam Reconstruction
3D Cone Beam Reconstruction

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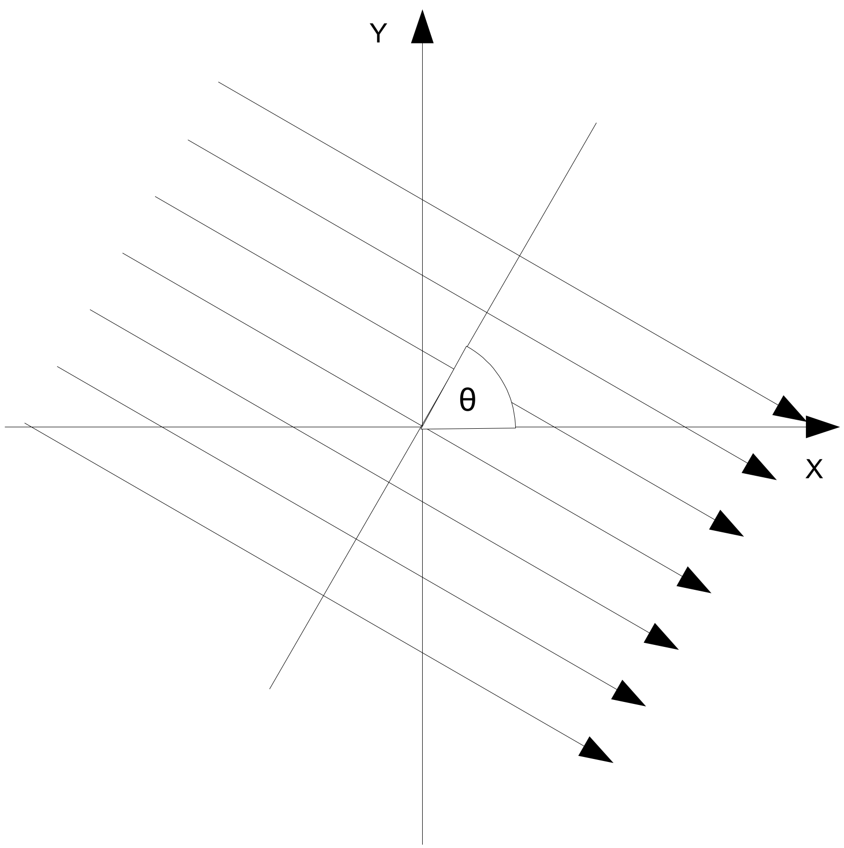
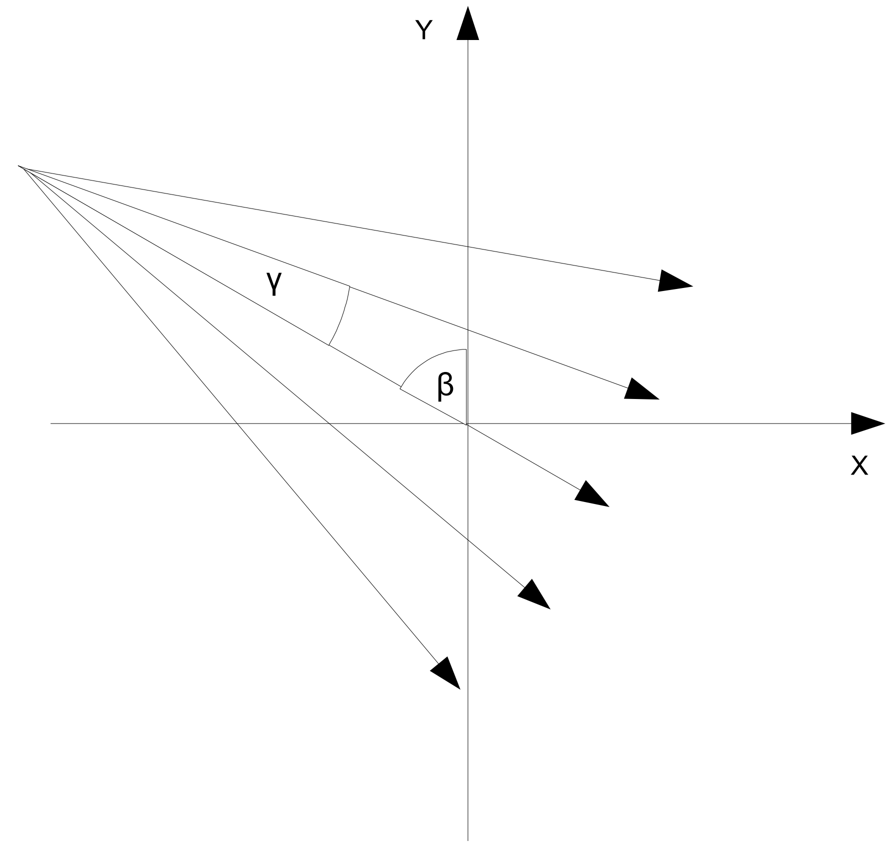
Overview

- 2D Fan Beam Reconstruction
- Shortscan Reconstruction
- 3D Cone Beam Reconstruction

Parallel vs. Fan-beam





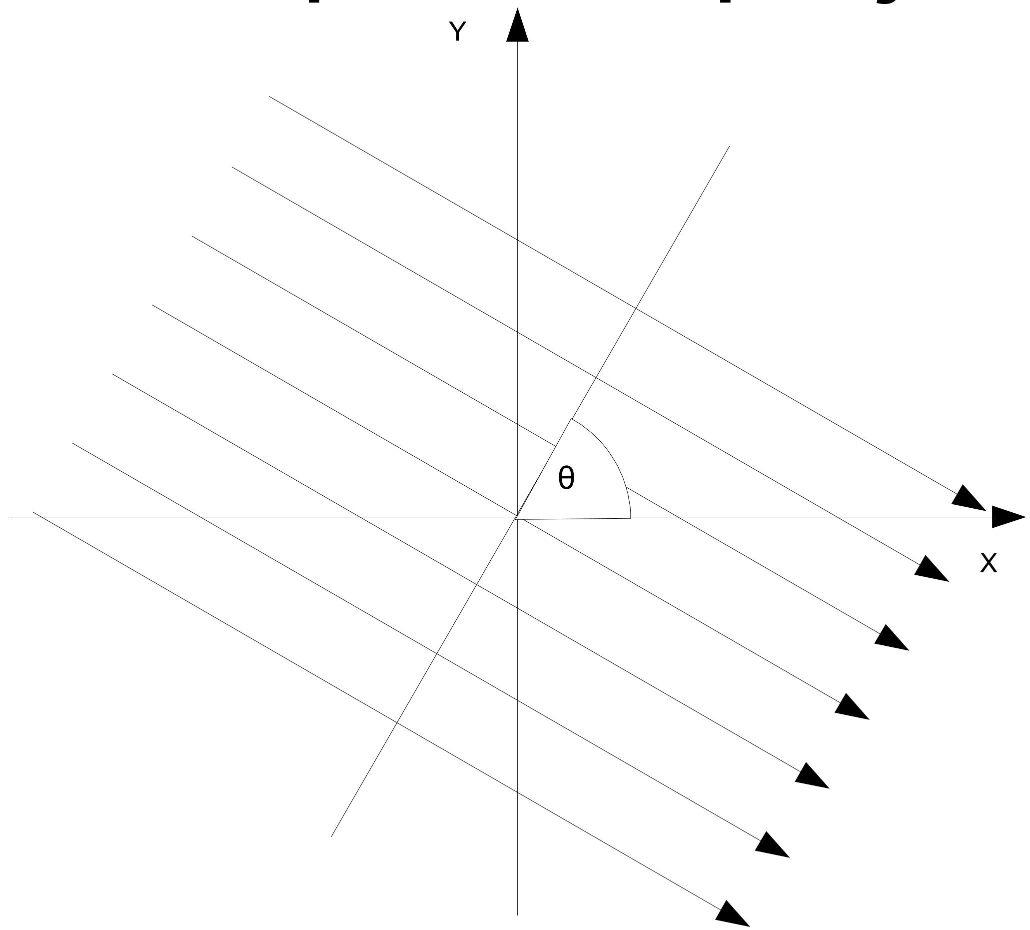
Measured projections:

$$P_{\theta}(t)$$

$$R_{\beta}(\gamma)$$

(R is measured at equiangular intervals)

FBP for parallel projections



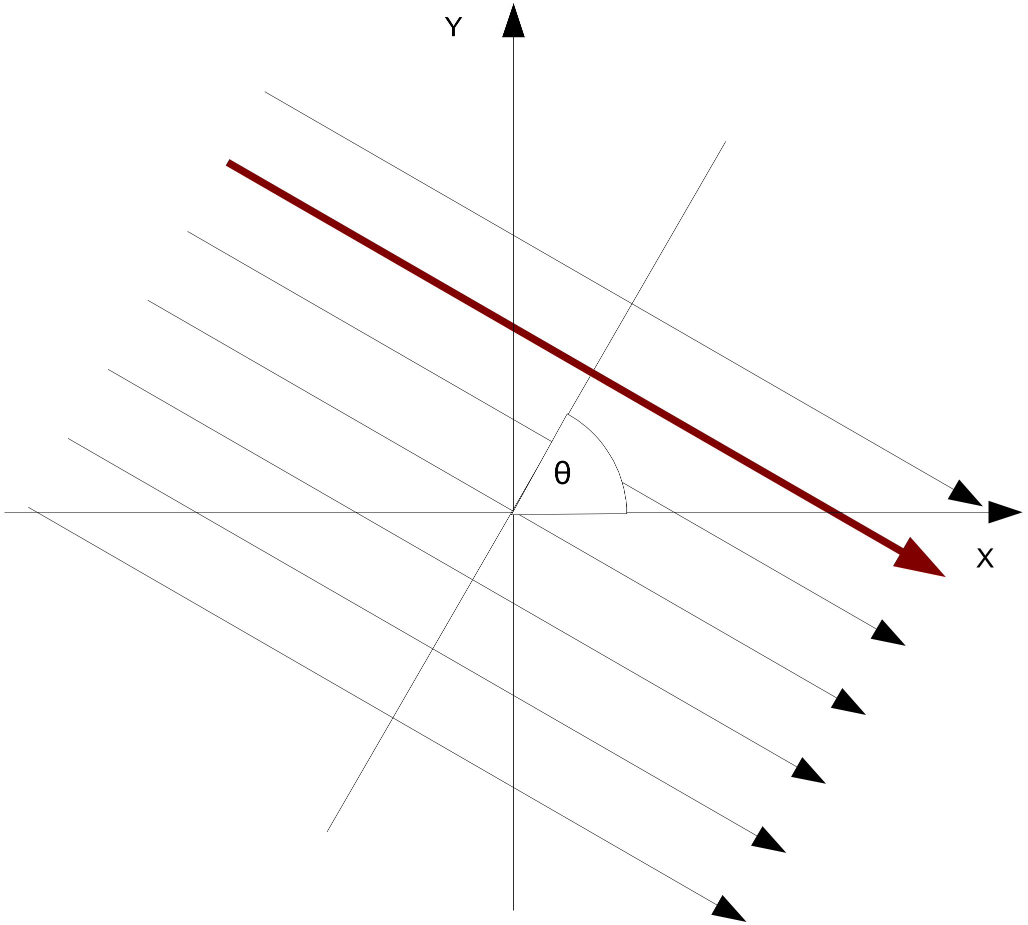
$$f(x,y)=\frac{1}{2}\int_{0}^{2\pi}\int_{-t_m}^{t_m} P_\theta(t)\,h(x\cos\theta+y\sin\theta-t)\,dt\,d\theta$$

$$f(r,\phi)=\frac{1}{2}\int_{0}^{2\pi}\int_{-t_m}^{t_m} P_\theta(t)\,h(r\cos(\theta-\phi)-t)\,dt\,d\theta$$

Polar coordinates:

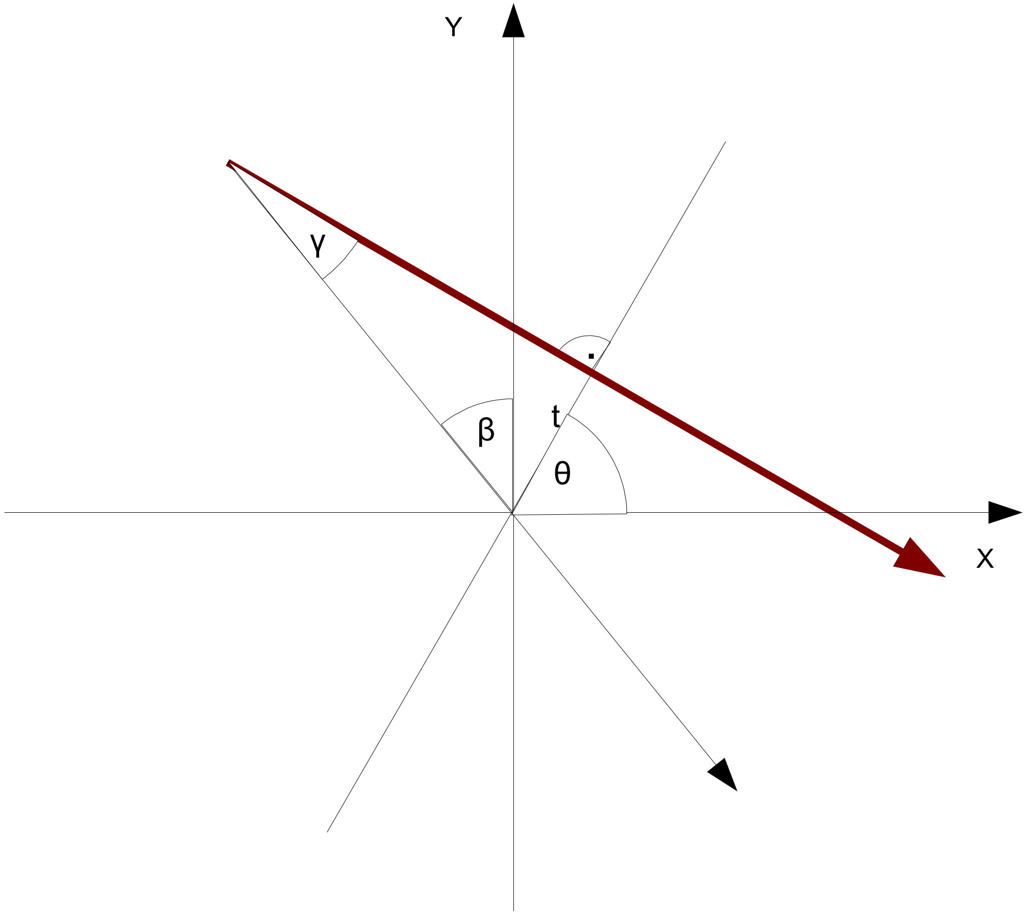
$$x=r\cos\phi$$
$$y=r\sin\phi$$

Transformation to Fan-beam geom.



Where can we find the data for this projection ray in the fan beam projection data?

Transformation to Fan-beam geom.



Transformation between θ,t and β,γ can be described by

$$\theta = \beta + \gamma$$
$$t = D \sin \gamma$$

Transformation of the FBP integral

$$f(r,\phi)=\frac{1}{2}\int_{0}^{2\pi}\int_{-t_m}^{t_m} P_\theta(t)\,h(r\cos(\theta-\phi)-t)\,dt\,d\theta$$

$$\theta=\beta+\gamma$$
$$t=D\sin\gamma$$
$$dt\,d\theta=D\cos\gamma\,d\gamma\,d\beta$$

$$=\frac{1}{2}\int_{-\gamma}^{2\pi-\gamma}\int_{-\sin^{-1}(t_m/D)}^{\sin^{-1}(t_m/D)} P_{\beta+\gamma}(D\sin\gamma)\,h(r\cos(\beta+\gamma-\phi)-D\sin\gamma)\,D\cos\gamma\,d\gamma\,d\beta$$

$$=\frac{1}{2}\int_{0}^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,h(r\cos(\beta+\gamma-\phi)-D\sin\gamma)\,D\cos\gamma\,d\gamma\,d\beta$$

Transformation of the FBP integral

$$\theta = \beta + \gamma$$
$$t = D\sin\gamma$$
$$dt\,d\theta = D\cos\gamma\,d\gamma\,d\beta$$

$$f(r,\phi) = \frac{1}{2}\int_{0}^{2\pi}\int_{-t_m}^{t_m} P_\theta(t)\,h(r\cos(\theta-\phi)-t)\,dt\,d\theta$$

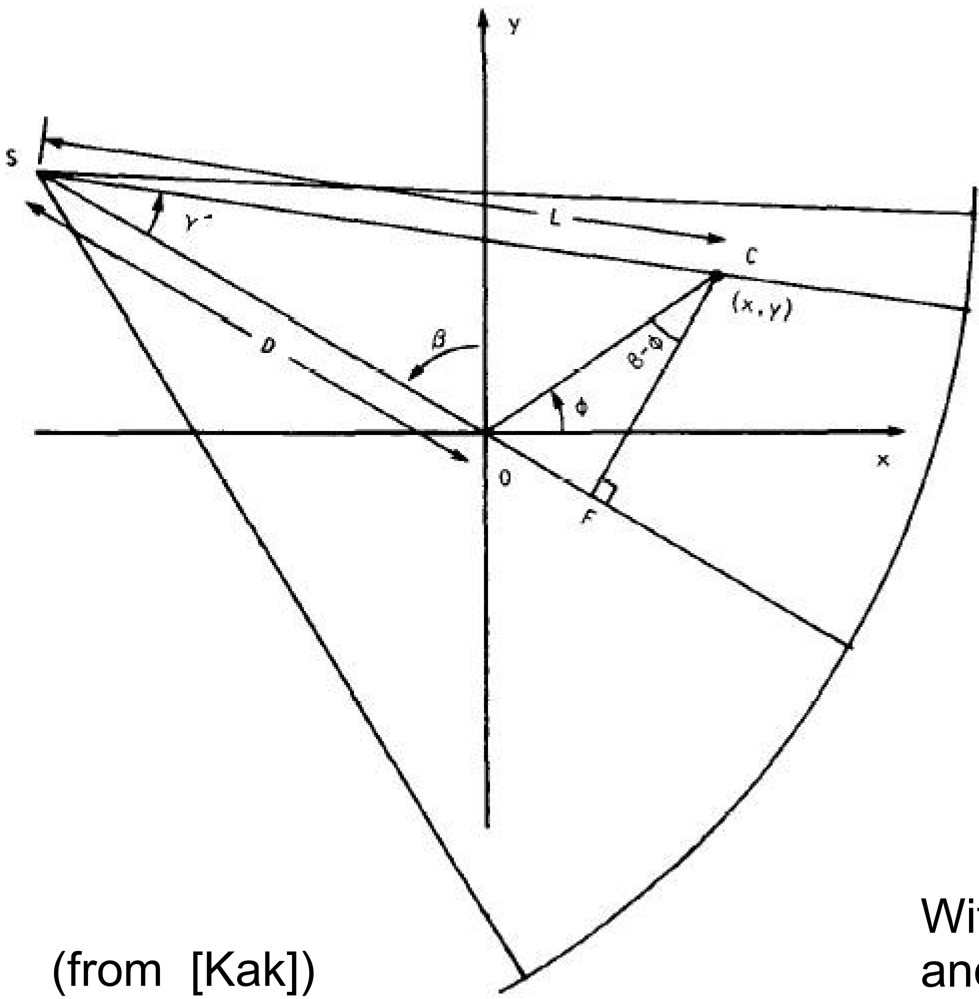
$$= \frac{1}{2}\int_{-\gamma}^{2\pi-\gamma}\int_{-\sin^{-1}(t_m/D)}^{\sin^{-1}(t_m/D)} P_{\beta+\gamma}(D\sin\gamma)\,h(r\cos(\beta+\gamma-\phi)-D\sin\gamma)\,D\cos\gamma\,d\gamma\,d\beta$$

$$= \frac{1}{2}\int_{0}^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,h(r\cos(\beta+\gamma-\phi)-D\sin\gamma)\,D\cos\gamma\,d\gamma\,d\beta$$

Further simplification can be applied to the argument of the function h.

Transformation of the FBP integral

Consider a point C in polar coordinates



(from [Kak])

$$L\cos\gamma' = D + r\sin(\beta - \phi)$$

$$L\sin\gamma' = r\cos(\beta - \phi)$$

$$r\cos(\beta+\gamma-\phi) - D\sin\gamma$$
$$= r\cos(\beta-\phi)\cos\gamma - [r\sin(\beta-\phi)+D]\sin\gamma$$
$$= L\sin\gamma'\cos\gamma - L\cos\gamma'\sin\gamma$$
$$= L\sin(\gamma'-\gamma)$$

With L being the distance from C to the source S and γ' being the angle of the projection ray C lies on

Transformation of the FBP integral

h(L sin γ) can be expressed in terms of h(γ):

$$h(t)=\int_{-\infty}^{\infty}|\omega|\exp(2\pi i\omega t)\,d\omega$$

$$h(L\sin\gamma)=\int_{-\infty}^{\infty}|\omega|\exp(2\pi i\omega L\sin\gamma)\,d\omega$$

Transformation: $\omega'=\dfrac{\omega L\sin\gamma}{\gamma}$

$$=\left(\frac{\gamma}{L\sin\gamma}\right)^2\int_{-\infty}^{\infty}|\omega'|\exp(2\pi i\omega'\gamma)\,d\omega$$

$$=\left(\frac{\gamma}{L\sin\gamma}\right)^2 h(\gamma)=\frac{2}{L^2}\frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 h(\gamma)$$

$$=:\frac{2}{L^2}g(\gamma)$$

Putting things together

$$f(r,\phi)=\frac{1}{2}\int_0^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,h(r\cos(\beta+\gamma-\phi)-D\sin\gamma)\,D\cos\gamma\,d\gamma\,d\beta$$

$$=\frac{1}{2}\int_0^{2\pi}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,h(L\sin(\gamma'-\gamma))\,D\cos\gamma\,d\gamma\,d\beta$$

$$=\int_0^{2\pi}\frac{1}{L^2}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,g(\gamma'-\gamma)\,D\cos\gamma\,d\gamma\,d\beta \qquad R'_\beta(\gamma):=R_\beta(\gamma)\,D\cos\gamma$$

$$=\int_0^{2\pi}\frac{1}{L^2}\int_{-\gamma_m}^{\gamma_m} R'_\beta(\gamma)\,g(\gamma'-\gamma)\,d\gamma\,d\beta \qquad Q_\beta(\gamma):=R'_\beta(\gamma)*g(\gamma)$$

$$=\int_0^{2\pi}\frac{1}{L^2}Q_\beta(\gamma')\,d\beta$$

The algorithm

$$f(r,\phi)=\int_0^{2\pi}\frac{1}{L^2}\int_{-\gamma_m}^{\gamma_m} R_\beta(\gamma)\,D\cos\gamma\; g(\gamma'-\gamma)\,d\gamma\, d\beta$$

Step 1: Preprocess each projection using $R'_\beta(\gamma):=R_\beta(\gamma)\,D\cos\gamma$

Step 2: Convolve each projection with the impulse response

$$g(\gamma)=\frac{1}{2}\left(\frac{\gamma}{\sin\gamma}\right)^2 h(\gamma)$$

Step 3: Perform a weighted backprojection of each filtered projection along the fan,

Where L is the distance of a given point (r, Φ) from the source:

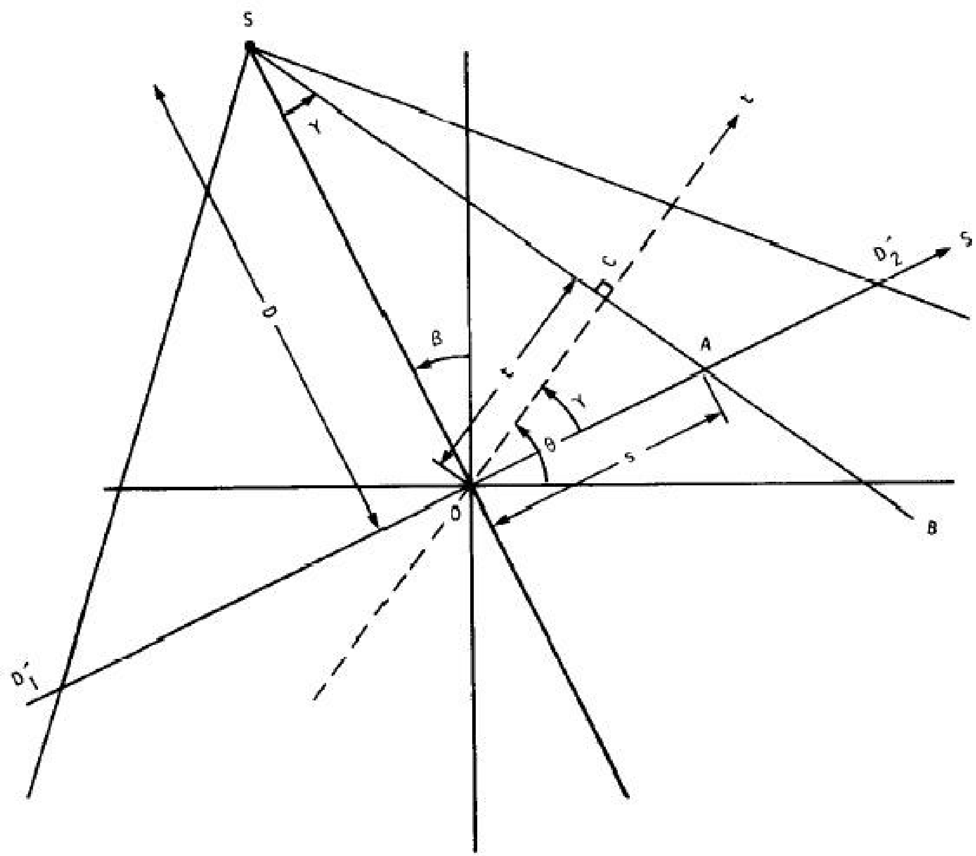
$$L(r,\phi,\beta)=\sqrt{[D+r\sin(\beta-\phi)]^2+[r\cos(\beta-\phi)]^2}$$

And γ' can be computed according to

$$\gamma'=\tan^{-1}(r\cos(\beta-\phi)/(D+r\sin(\beta-\phi)))$$

Equally spaced collinear detector

So far: we used projection rays on equiangular intervals
Now: use equally spaced projections on a (virtual) collinear detector



(from [Kak])

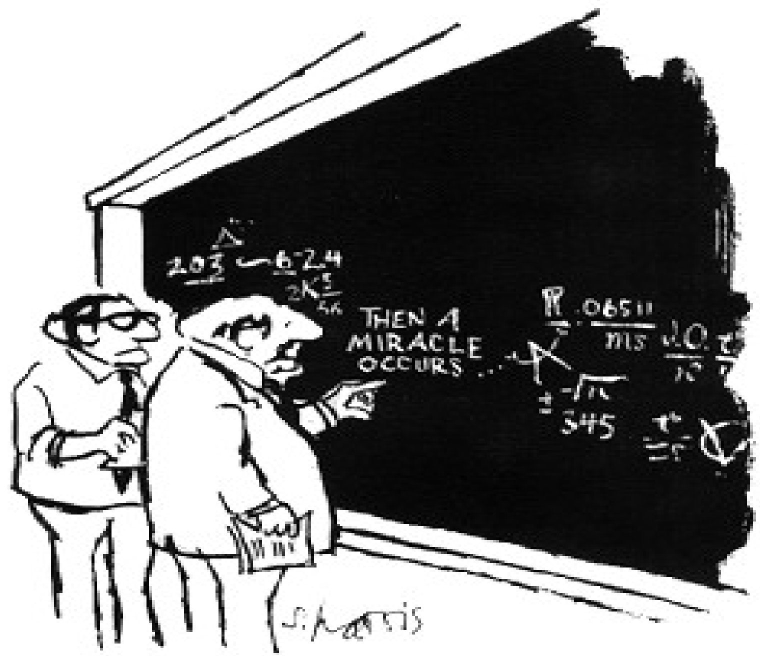
Transformation between fan-beam data and parallel data:

$$\theta = \beta + \gamma$$

$$t = s \cos\gamma = \frac{s\,D}{\sqrt{D^2 + s^2}}$$

Equally spaced collinear detector

Similar to the equiangular case, we can transform the parallel FBP to β, s



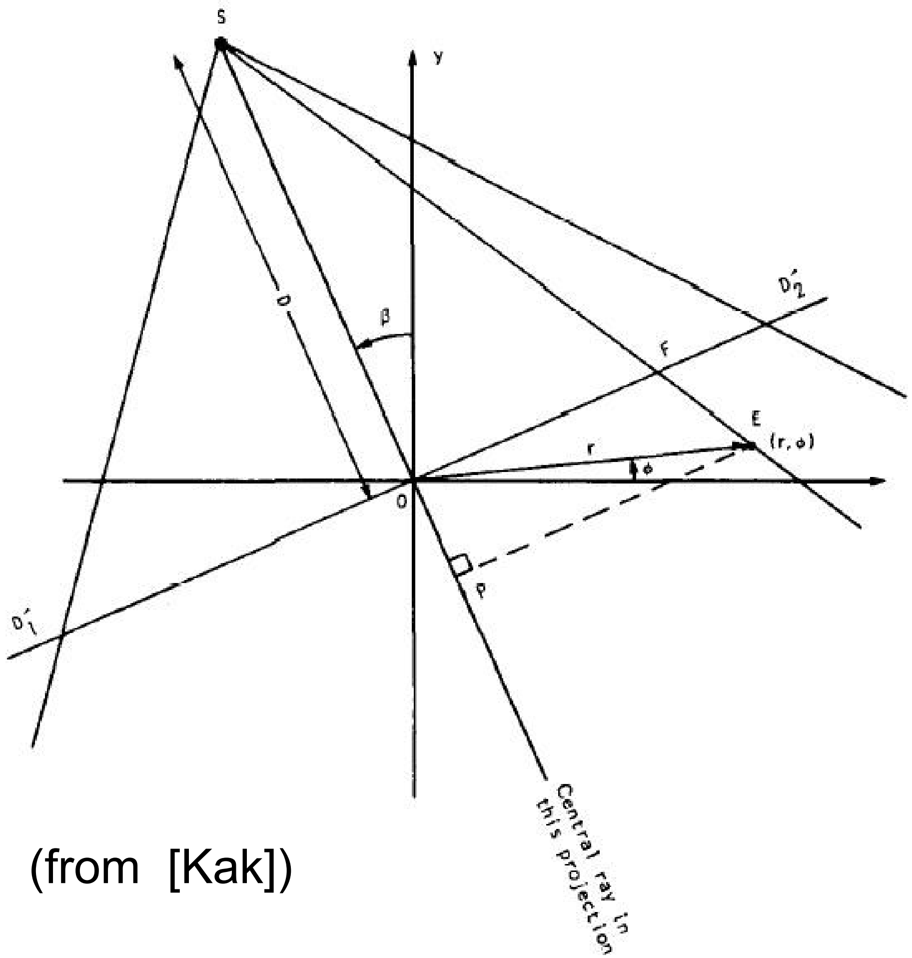
"I think you should be more explicit here in step two."

... and end up with the FBP formula for the collinear detector:

$$f(r,\phi)=\int_0^{2\pi}\frac{1}{U^2}\int_{-\infty}^{\infty}R_\beta(s)\frac{D}{\sqrt{D^2+s^2}}\frac{h(s'-s)}{2}\,ds\,d\beta$$

Equally spaced collinear detector

$$f(r,\phi)=\int_0^{2\pi}\frac{1}{U^2}\int_{-\infty}^{\infty}R_\beta(s)\frac{D}{\sqrt{D^2+s^2}}\frac{h(s'-s)}{2}\,ds\,d\beta$$



(from [Kak])

Where s' is the backprojected coordinate of (r,Φ) in Projection β

$$s'=\frac{\bar{EP}}{\bar{SP}}\bar{SO}=D\frac{r\cos(\beta-\phi)}{D+r\sin(\beta-\phi)}$$

And U is the ratio

$$U(r,\phi,\beta)=\frac{\bar{SO}+\bar{OP}}{D}=\frac{D+r\sin(\beta-\phi)}{D}$$

of the projection onto the central ray.

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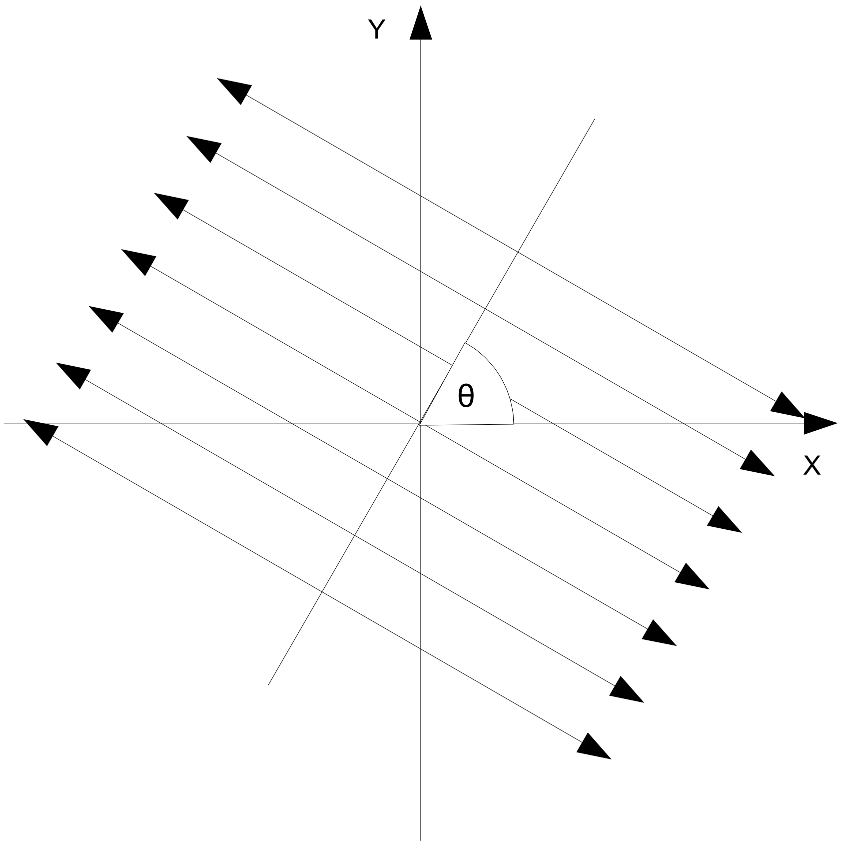
Parallel beam shortscan

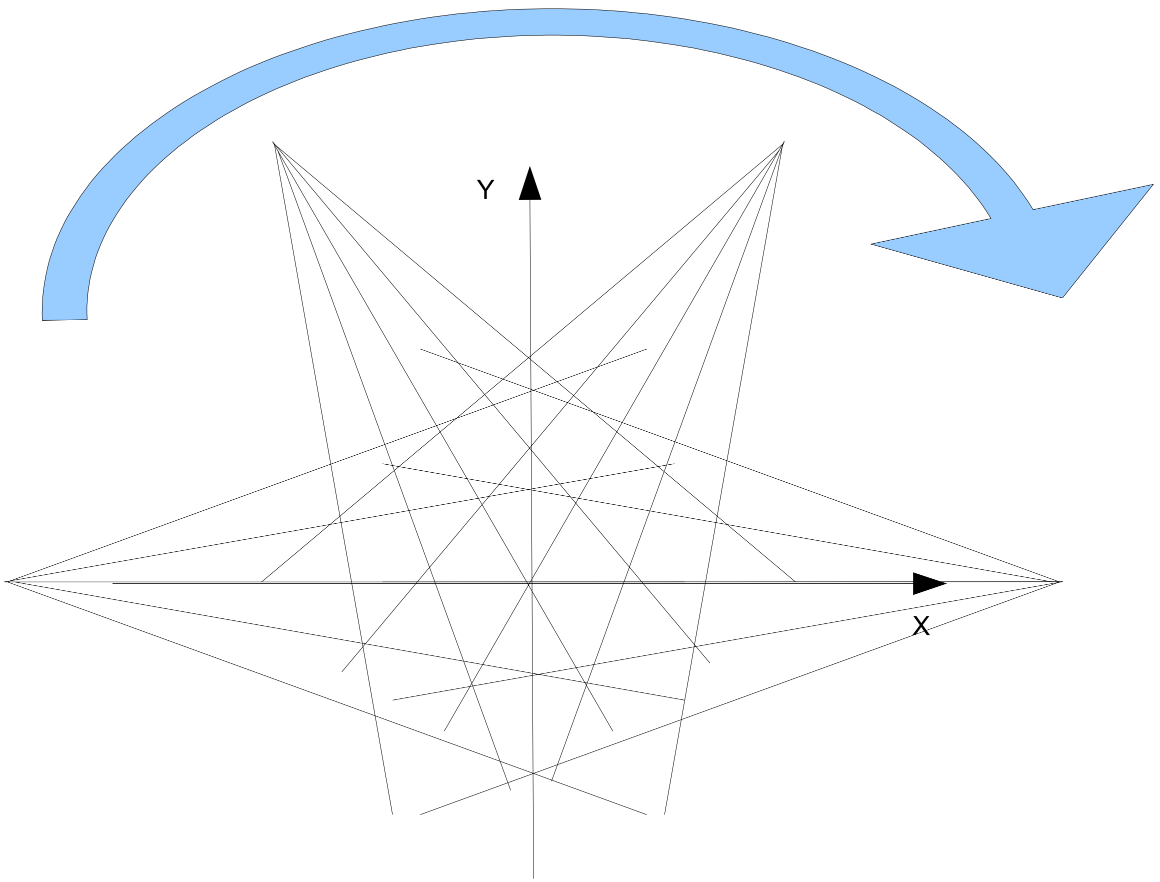
For the case of parallel projections, it is easy to see that a scan over an angle of 180° is sufficient, because projections that are 180° apart, are mirror images of each other:



$$P_\theta(t) = P_{\theta+180^\circ}(-t)$$

Fan-beam shortscan

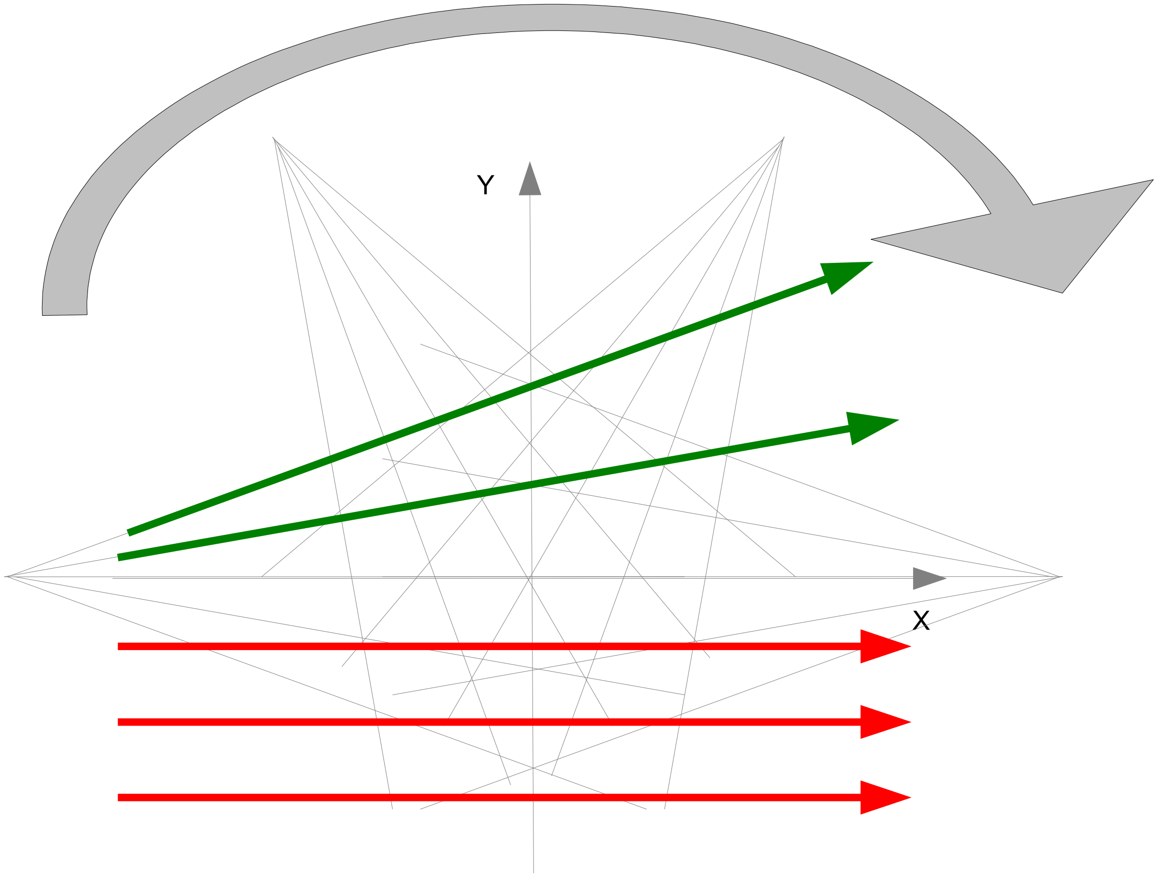
Consider a Fan-beam scan over 180°:



Question: is the collected data sufficient for reconstruction?

Fan-beam shortscan

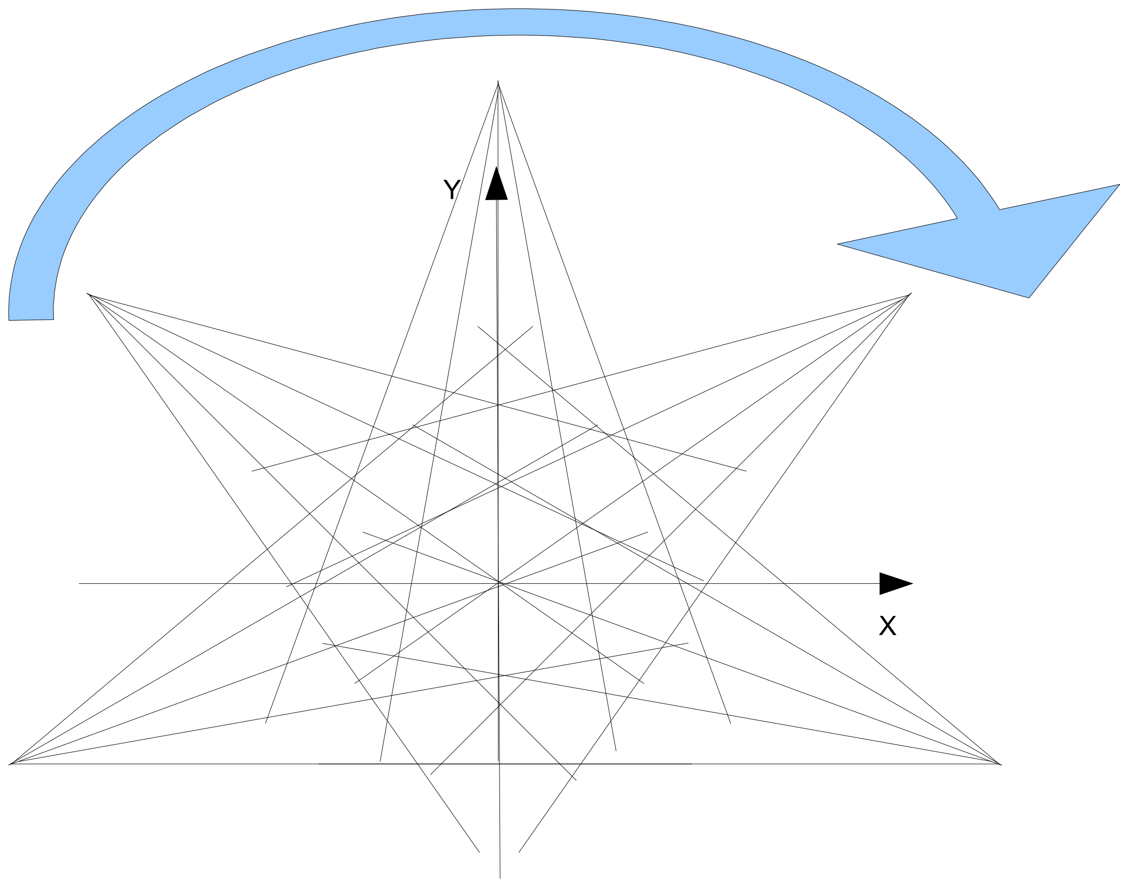
Consider a Fan-beam scan over 180°:



No! Some projection rays are missing in the measured data,
other rays are measured twice, what leads to redundancy in the data set.

Fan-beam shortscan

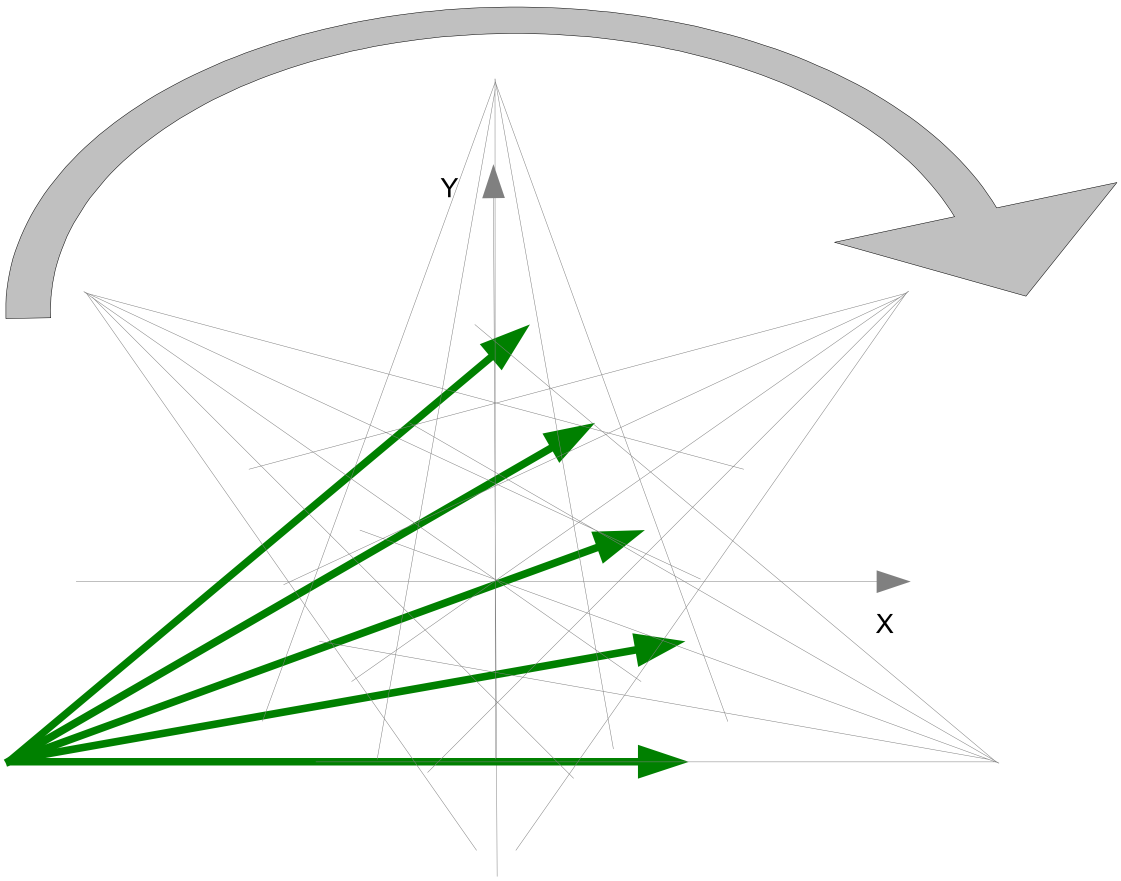
To be able to reconstruct an object, we need to scan at least over an interval of $180° + 2\gamma_m$



Unfortunately, this adds even more redundancy to our measurement.

Fan-beam shortscan

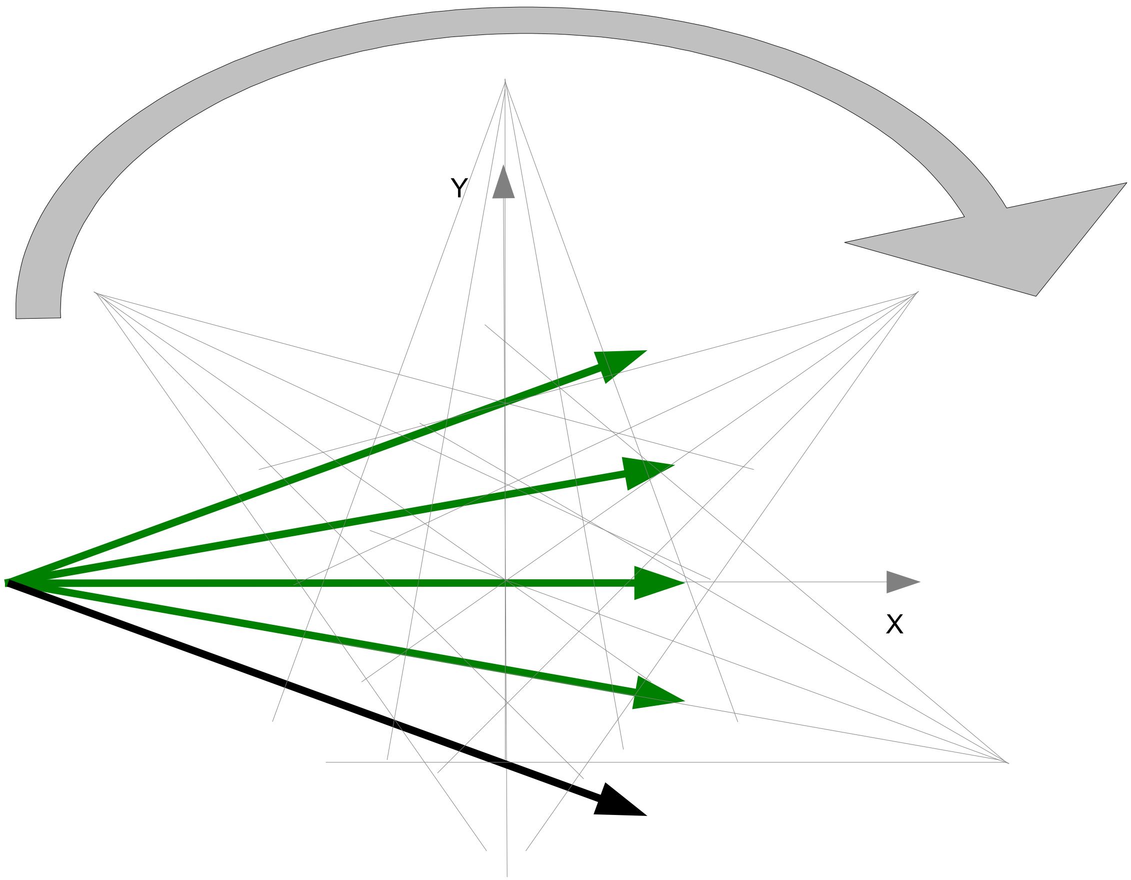
To be able to reconstruct an object, we need to scan at least over an interval of $180°+2\gamma_m$



β	*redundant range for* γ
0	$-\gamma_m \le \gamma \le \gamma_m$

Fan-beam shortscan

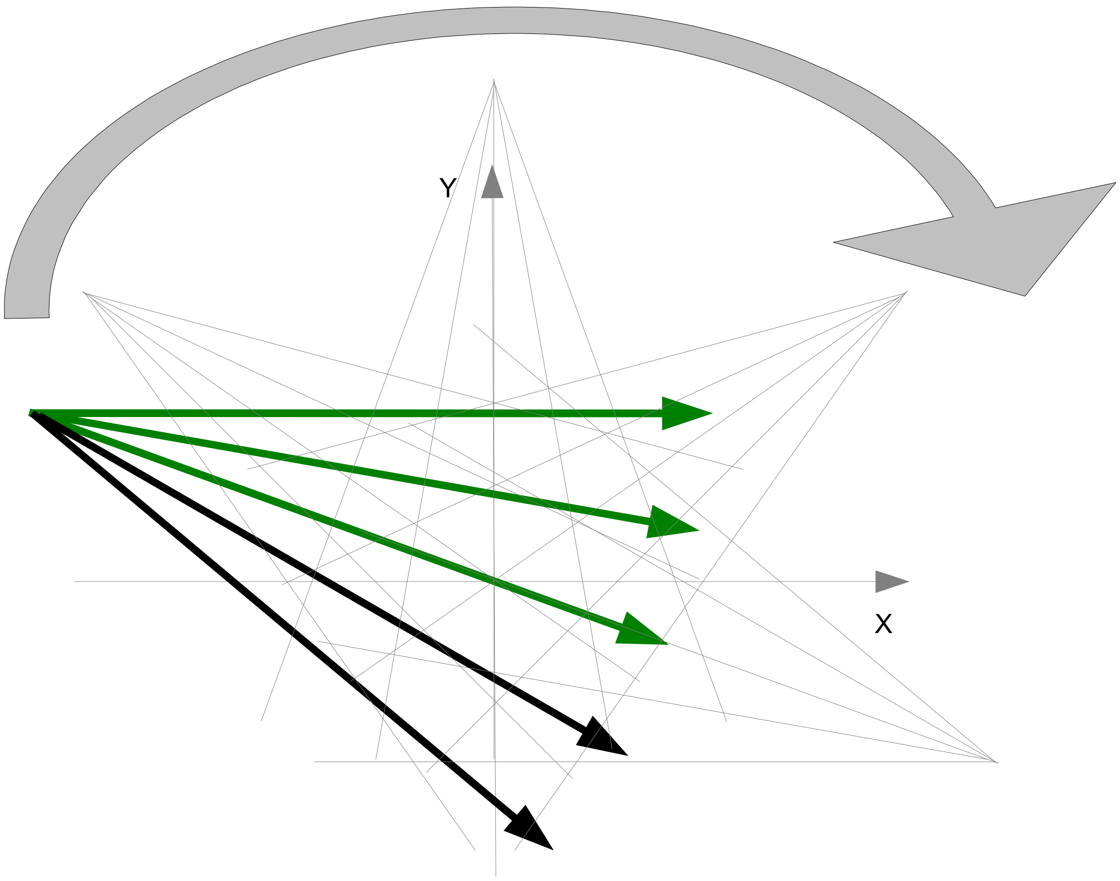
To be able to reconstruct an object, we need to scan at least over an interval of $180° + 2\gamma_m$



β	*redundant range for* γ
0	$-\gamma_m \leq \gamma \leq \gamma_m$
γ_m	$-\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$

Fan-beam shortscan

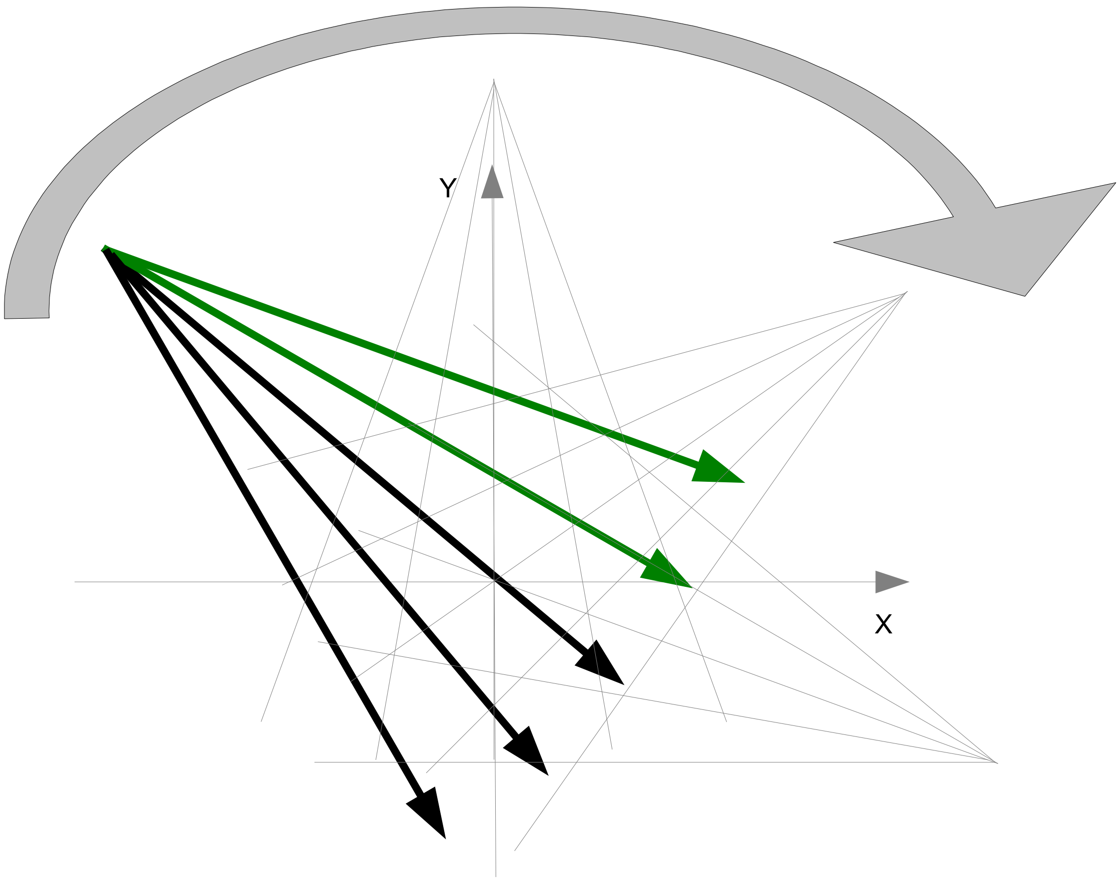
To be able to reconstruct an object, we need to scan at least over an interval of $180°+2\gamma_m$



β	*redundant range for* γ
0	$-\gamma_m \leq \gamma \leq \gamma_m$
γ_m	$-\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$2\gamma_m$	$0 \leq \gamma \leq \gamma_m$

Fan-beam shortscan

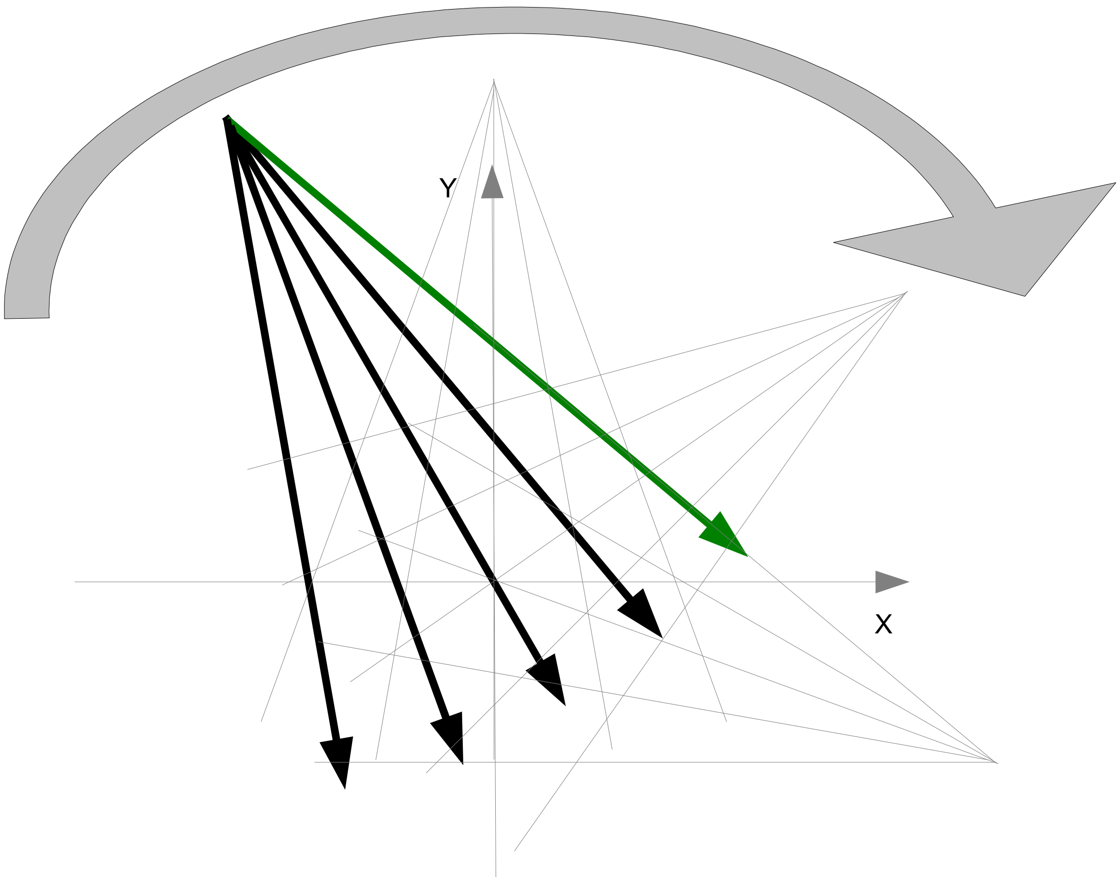
To be able to reconstruct an object, we need to scan at least over an interval of $180°+2\gamma_m$



β	*redundant range for* γ
0	$-\gamma_m \leq \gamma \leq \gamma_m$
γ_m	$-\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$2\gamma_m$	$0 \leq \gamma \leq \gamma_m$
$3\gamma_m$	$\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$

Fan-beam shortscan

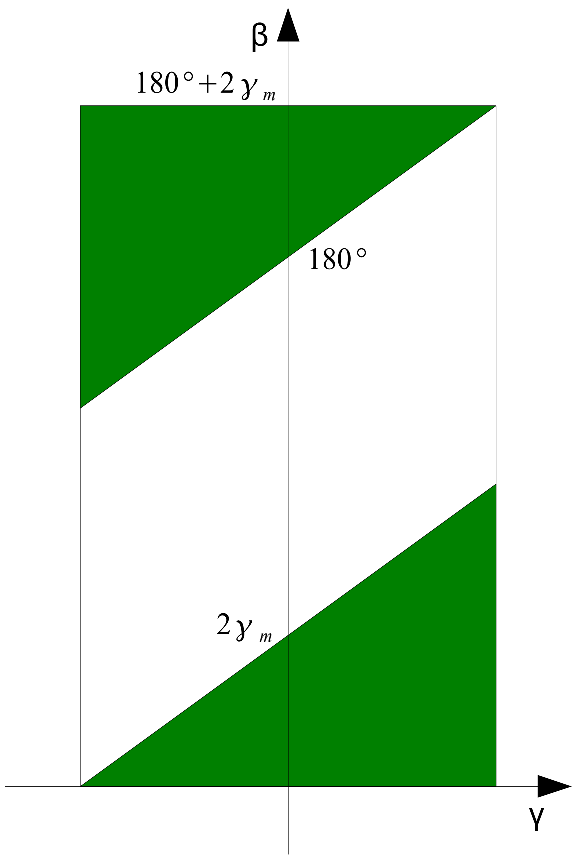
To be able to reconstruct an object, we need to scan at least over an interval of $180° + 2\gamma_m$



β	*redundant range for* γ
0	$-\gamma_m \leq \gamma \leq \gamma_m$
γ_m	$-\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$2\gamma_m$	$0 \leq \gamma \leq \gamma_m$
$3\gamma_m$	$\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$4\gamma_m$	$\gamma = \gamma_m$

Sinogram

The sinogram shows the value of the line integral for a specific β,γ

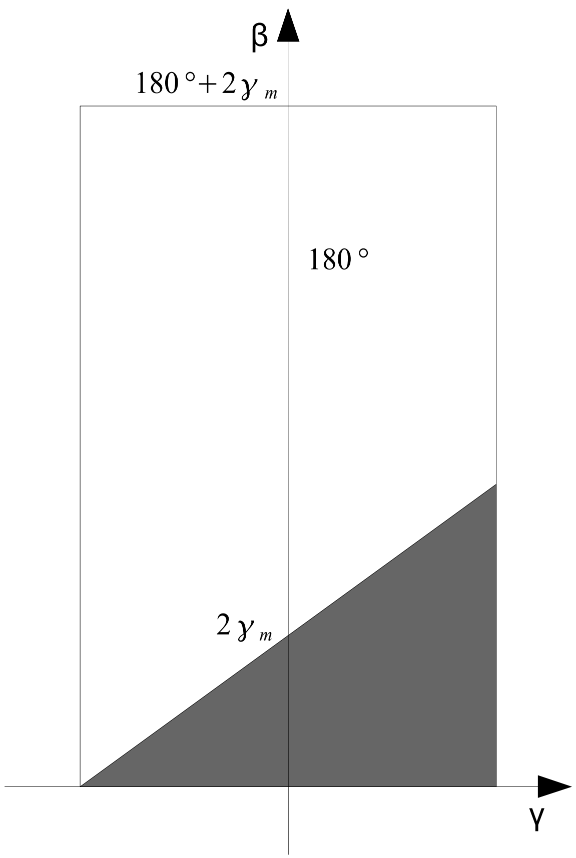


β	*redundant range for* γ
0	$-\gamma_m \leq \gamma \leq \gamma_m$
γ_m	$-\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$2\gamma_m$	$0 \leq \gamma \leq \gamma_m$
$3\gamma_m$	$\frac{1}{2}\gamma_m \leq \gamma \leq \gamma_m$
$4\gamma_m$	$\gamma = \gamma_m$

The green area shows the data that is measured twice

Handling redundancy

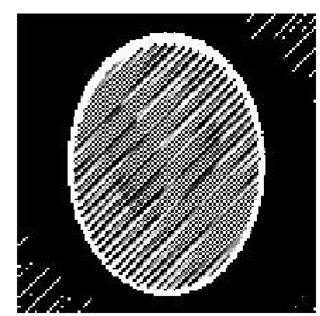
1st try: use a binary window function to hide redundant data



$$w_\beta(\gamma) = \begin{cases} 0 & 0 \le \beta \le 2\gamma_m + 2\gamma \\ 1 & elsewhere \end{cases}$$

This window function is not usable, because the sharp edge introduces high frequencies, that causes strong artifacts during the filtering process:

Full scan

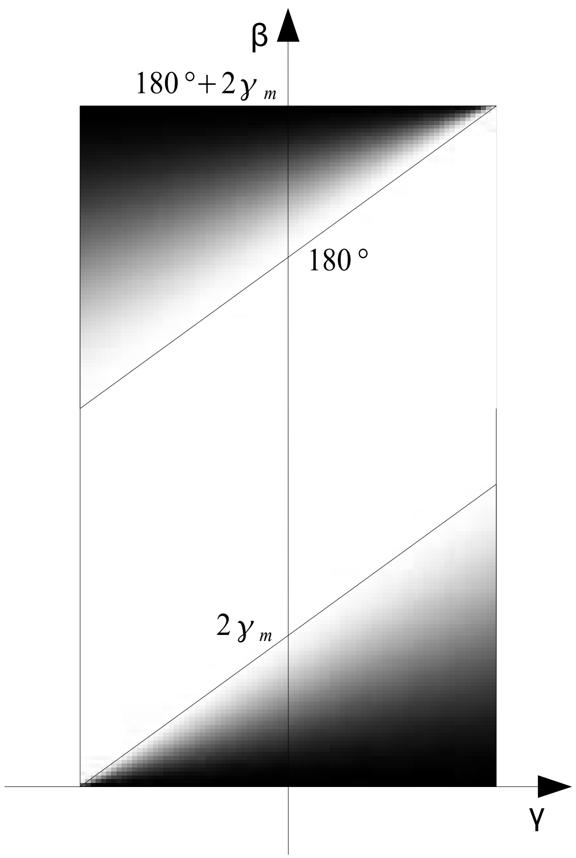


Short scan with binary window

(from [Tur])

Handling redundancy

2nd try: use a smooth window function to hide redundant data: „Parker weighting“



$$w_\beta(\gamma)=\begin{cases} \sin^2\left(45°\dfrac{\beta}{\gamma_m+\gamma}\right) & 0\le\beta\le 2\gamma_m+2\gamma \\ \sin^2\left(45°\dfrac{2\gamma_m+180°-\beta}{\gamma_m-\gamma}\right) & 180°+2\gamma\le\beta\le 180°+2\gamma_m \\ 1 & \text{elsewhere} \end{cases}$$

Some properties:

w is continuously differentiable

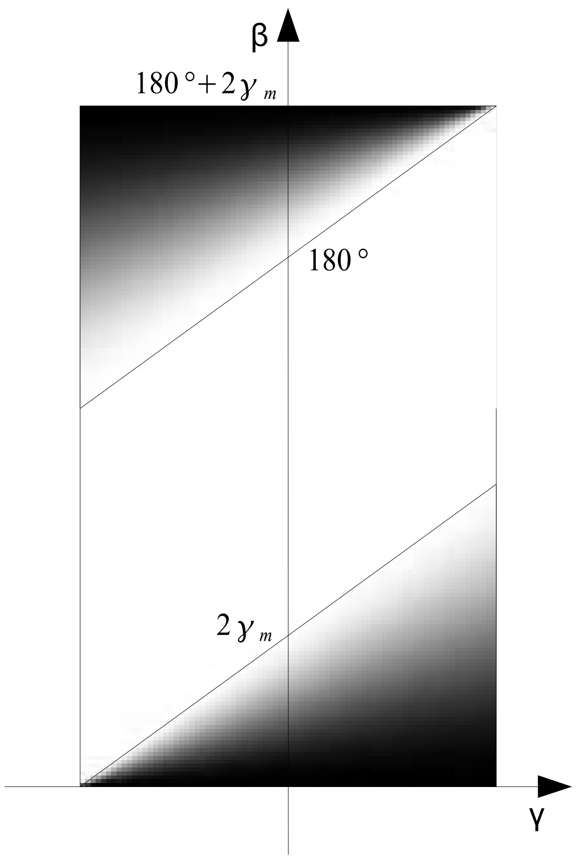
$w_{\beta_1}(\gamma_1)+w_{\beta_2}(\gamma_2)=1$ for redundant rays

$w_\beta(\gamma)=0$ for $\beta=0$ and $\beta=180°+2\gamma_m$

$\dfrac{\partial w_\beta(\gamma)}{\partial\beta}=0$ for $\beta=2\gamma_m+2\gamma$ and $\beta=180°+2\gamma$

Handling redundancy

2^{nd} try: use a smooth window function to hide redundant data: „Parker weighting“



$$w_\beta(\gamma)=\begin{cases} \sin^2\left(45°\dfrac{\beta}{\gamma_m+\gamma}\right) & 0\le\beta\le 2\gamma_m+2\gamma \\ \sin^2\left(45°\dfrac{2\gamma_m+180°-\beta}{\gamma_m-\gamma}\right) & 180°+2\gamma\le\beta\le 180°+2\gamma_m \\ 1 & \textit{elsewhere} \end{cases}$$

Some results:

Full scan

Short scan with Parker weighting

(from [Tur])

Parallel rebinning

Reconstruction from Fan-beam geometry can also be done by re-sorting the fan-beam data into a parallel dataset:

$$P_\theta(t) = R_{\theta - \sin^{-1}\frac{t}{D}}\left(\sin^{-1}\frac{t}{D}\right)$$

using

$$\theta = \beta + \gamma$$
$$t = D\sin\gamma$$

and applying the FBP for the parallel case to this dataset.

Problem: for discrete functions P and R this requires interpolation

Overview

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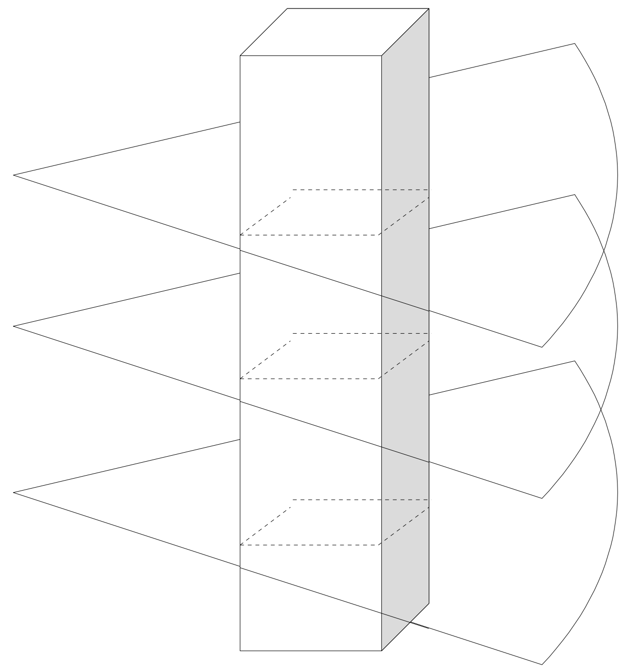
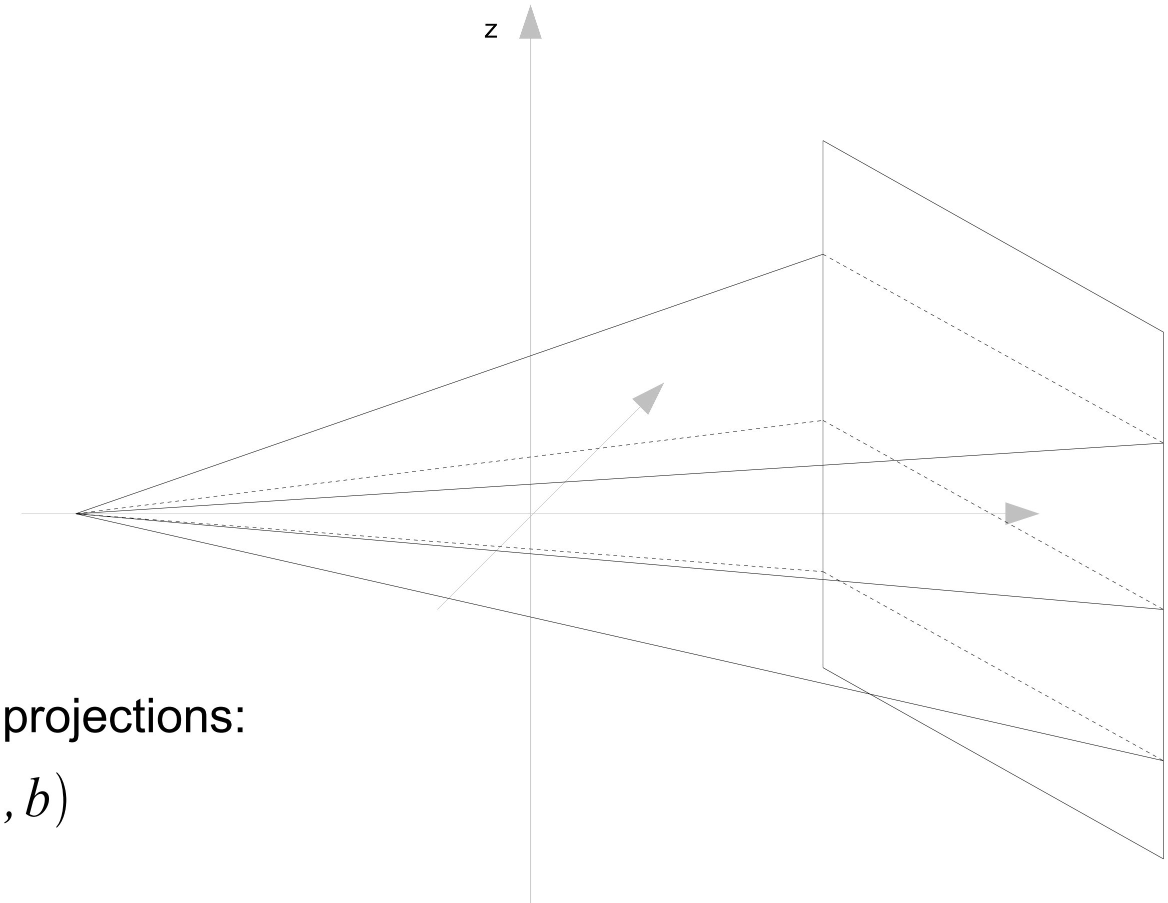
3D Reconstruction

A 3D object can be reconstructed using 2D methods by scanning it slice by slice:



- Requires a complex mechanical system
- The scan is very time-consuming, because the X-Ray source or the object has to be moved in every step
- The X-Ray dose is not used optimally

3D Cone Beam Reconstruction



Measured projections:

$$R_\beta(a,b)$$

- Multiple fans with the same origin
- The X-ray source is rotated on a circular trajectory in the x/y plane
- The detector has planar shape
- Using this geometry, it is not possible to do an exact reconstruction of the object

The Feldkamp Algorithm

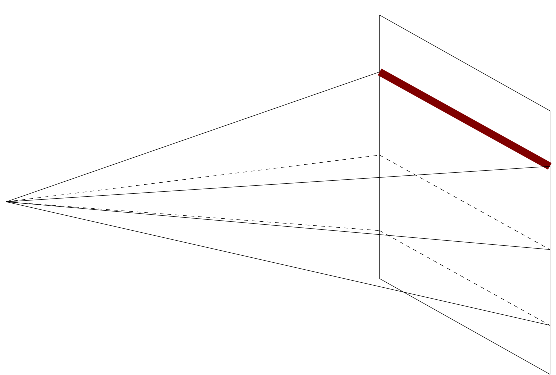
- Developed in 1984 by Feldkamp, Davis, Kress
- Approximate method for reconstruction from Cone beam data
- The reconstruction is based on filtering and backprojecting the single planes in the cone independently.
- The reconstruction result is obtained by summing up the contributions of all tilted fan beams.
- The final algorithm is very similar to the 2D algorithm for equispaced fan beams on a collinear detector.

The Feldkamp Algorithm

Step 1: Preprocess each projection using

$$R'_\beta(a,b) := R_\beta(a,b)\frac{D}{\sqrt{D^2+a^2+b^2}}$$

Step 2: Convolve each projection in 'a'-direction with the impulse response



$$g(a) = \frac{1}{2}h(a)$$

$$Q_\beta(a,b) = R'_\beta(a,b) * \frac{1}{2}h(a)$$

The Feldkamp Algorithm

Step 3: Perform a weighted backprojection of each filtered projection along the cone

$$f(x,y,z)=\int_0^{2\pi} \frac{1}{U^2} Q_\beta(a(x,y,\beta),b(x,y,z,\beta))$$

Where a and b can be found by backprojecting voxel (x,y,z) into projection β:

$$a(x,y,\beta)=D\frac{x\cos\beta+y\sin\beta}{D+x\sin\beta-y\cos\beta}$$

$$b(x,y,z,\beta)=\frac{zD}{D+x\sin\beta-y\cos\beta}$$

And the weighting factor U is identical to the factor for fan beam reconstruction:

$$U(x,y,\beta)=\frac{D+x\sin\beta-y\cos\beta}{D}$$

Some notes

- The Feldkamp algorithm is exact for points in the x/y plane (the plane where the source trajectory lies).

- The computation of the coordinates in the projections of a backprojected voxel can be rewritten to a linear mapping in homogeneous coordinates.

References

[Kak] A. C. Kak and Malcolm Slaney, Principles of Computerized Tomographic Imaging. Society of Industrial and Applied Mathematics, 2001

[Tur] Henrik Turbell, Cone-Beam Reconstruction using Filtered Backprojection. Dissertation, Linköping Studies in Science and Technology, 2001