

RESOURCE BOUNDED IMMUNITY AND SIMPLICITY *

Extended Abstract

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Abstract Revisiting the thirty years-old notions of resource-bounded immunity and simplicity, we investigate the structural characteristics of various immunity notions: strong immunity, almost immunity, and hyperimmunity as well as their corresponding simplicity notions. We also study k -immunity and k -simplicity and their extensions: feasible k -immunity and feasible k -simplicity. Finally, we propose the k -immune hypothesis as a working hypothesis that ensures the existence of simple sets in NP.

Keywords: immune set, simple set, complete set, forcing, generic oracle, random oracle

1. Foundations of Immunity and Simplicity

The original notions of immunity and simplicity date back to mid 1940s. Post [17] first constructed a simple set for the class of recursively enumerable sets. The new breed of resource-bounded immunity and simplicity waited to be introduced until mid 1970s by an early work of Flajolet and Steyaert [5]. In their seminal paper, Flajolet and Steyaert constructed various recursive sets that, for instance, have no infinite $\text{DTIME}(t(n))$ -subsets under the term “ $\text{DTIME}(t(n))$ -immune sets.” Later, Ko and Moore [11] studied the polynomial-time bounded immunity, which is now preferably called P-immune sets. Subsequently, Balcázar and Schöning [2] considered P-bi-immune sets, which are P-immune sets whose complements are also P-immune. Homer and Maass [8] extensively discussed the cousin of P-immune sets, known as NP-simple sets. The importance of these notions was widely recognized in 1980s. Since these notions can be easily expanded to any complexity class \mathcal{C} , we begin with

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an introduction of the general notions of \mathcal{C} -immune sets, \mathcal{C} -bi-immune sets, and \mathcal{C} -simple sets. These notions are further expanded in various manners in later sections.

DEFINITION 1 *Let \mathcal{C} be any complexity class of languages over alphabet Σ .*

- 1 *A set S is \mathcal{C} -immune if S is infinite and there is no infinite subset of S in \mathcal{C} .*
- 2 *A set S is \mathcal{C} -bi-immune if S and \bar{S} are both \mathcal{C} -immune.*
- 3 *A set S is \mathcal{C} -simple if S belongs to \mathcal{C} and \bar{S} is \mathcal{C} -immune.*

Note that the existence of a \mathcal{C} -simple set immediately implies $\mathcal{C} \neq \text{co-}\mathcal{C}$; however, the separation $\mathcal{C} \neq \text{co-}\mathcal{C}$ does not necessarily guarantee the existence of \mathcal{C} -simple sets.

Throughout this paper, we set our alphabet Σ to be $\{0, 1\}$. Let \mathbb{N} (or ω) denote the set of all nonnegative integers and set $\mathbb{N}^+ = \mathbb{N} - \{0\}$. All *logarithms* are taken to base 2 and a *polynomial* means a multivariate polynomial with integer coefficients. We assume a standard bijection from $\Sigma^{<\omega}$ to Σ^* that is polynomial-time computable and polynomial-time invertible, where $\Sigma^{<\omega}$ is the set of all finite sequences of strings over Σ . This bijection enables us to identify $\Sigma^{<\omega}$ with Σ^* . We use *multi-tape off-line Turing machines (TMs, in short)* as a model of computation. Assumed is the reader's familiarity with basic complexity classes, such as P, NP, E (linear exponential time), and EXP (polynomial exponential time). This paper focuses mostly on the complexity classes lying in the *polynomial-time hierarchy*¹ $\{\Delta_k^P, \Sigma_k^P, \Pi_k^P \mid k \in \mathbb{N}\}$ [12].

We mainly use "partial" functions and all functions are presumed to be single-valued. Since total functions are also partial functions, we will define function classes as collections of *partial* functions and, whenever we need total functions, we will explicitly indicate the *totality* of functions. Now, fix $k \in \mathbb{N}^+$. The notation $\text{F}\Delta_k^P$ denotes the collection of all single-valued partial functions f such that there exist a set $B \in \Sigma_{k-1}^P$ and a polynomial-time deterministic oracle TM M satisfying the following condition: for every x , if $x \in \text{dom}(f)$ then $M^B(x)$ halts in an accepting state and outputs $f(x)$ and otherwise, $M^B(x)$ halts in a rejecting state (in this case, $f(x)$ is *undefined*). In particular, write FP for $\text{F}\Delta_1^P$.

A set A is called Δ_k^P -*m-reducible* to another set B via a reduction f if f is a total $\text{F}\Delta_k^P$ -function from Σ^* to Σ^* and $A = \{x \mid f(x) \in B\}$. If in addition f is honest², then we say that A is h - Δ_k^P -*m-reducible* to B . Moreover, a set A is Δ_k^P -*tt-reducible* to B via a reduction triplet (ν, f, α) if (i) ν is a total $\text{F}\Delta_k^P$ -function from Σ^* to $\{0\}^*$, (ii) f is a total $\text{F}\Delta_k^P$ -function from Σ^* to Σ^* such that, for every x , $f(x) = \langle y_1, y_2, \dots, y_k \rangle$ for certain strings y_1, y_2, \dots, y_k , where $k = |\nu(x)|$, (iii) α is a total $\text{F}\Delta_k^P$ -function from $\Sigma^* \times \Sigma^*$ to $\{0, 1\}$ such that $A = \{x \mid \alpha(x, B(f(x))) = 1\}$, where $B(f(x))$ is an abbreviation of the k -bit string $B(y_1)B(y_2) \cdots B(y_k)$ when $f(x) = \langle y_1, \dots, y_k \rangle$. If in addition f is componentwise honest³, then A is h - Δ_k^P -*tt-reducible* to B . The notion of *completeness* can be induced from its corresponding reducibility.

¹The polynomial-time hierarchy consists of the classes defined in the following fashion: $\Delta_0^P = \Sigma_0^P = \Pi_0^P = \text{P}$, $\Sigma_{k+1}^P = \text{NP}^{\Sigma_k^P}$, and $\Pi_{k+1}^P = \text{co-}\Sigma_{k+1}^P$ for any number $k \in \mathbb{N}$.

²A partial function f from Σ^* to Σ^* is *polynomially honest* (*honest*, for short) if there exists a polynomial p such that $|x| \leq p(|f(x)|)$ for all strings $x \in \text{dom}(f)$.

³A partial function from Σ^* to $\Sigma^{<\omega}$ (which is identified with Σ^*) is *componentwise honest* if there exists a polynomial p such that, for every $x \in \text{dom}(f)$, $|x| \leq p(|y_i|)$ for all $i \in \{1, 2, \dots, k\}$, provided that $f(x) = \langle y_1, y_2, \dots, y_k \rangle$.

It is well-known that P-immune sets exist even in the class E. In particular, Ko and Moore [11] constructed a P-immune set that is also P-tt-complete for E. Note that no h-P-m-complete set for NP can be P-immune since the image of a P-immune set by a polynomial-time computable reduction is either finite or P-immune. Using a relativization technique, Bennett and Gill [3] showed that a P-immune set exists in NP relative to a random oracle with probability 1. A recursive oracle relative to which NP contains P-immune sets was later constructed by Homer and Maass [8]. Torenvliet and van Emde Boas [22] strengthened their results by demonstrating a relativized world where NP has a P-immune set which is also NP-simple.

The notion of \mathcal{C} -immunity is closely related to various other notions, which include complexity cores [13] and instance complexity [15]. We can naturally expand these characterizations to more general Σ_k^P -immune and Δ_k^P -immune sets. Balcázar and Schöning [2] also built a bridge between P-bi-immune sets and finite-to-one reductions. Expanding their argument, we give in Lemma 2 a characterization of \mathcal{C} -bi-immunity as well as \mathcal{C} -immunity.

For any partial function f from Σ^* to Σ^* , the set $Graph(f) = \{\langle x, f(x) \rangle \mid x \in \text{dom}(f)\}$ is called the *graph* of f . Let $\Sigma_0^P SV = FP$ and let $\Sigma_k^P SV$ denote the class of all single-valued partial functions f such that f is polynomially bounded⁴ and $Graph(f)$ is in Σ_k^P . For brevity, we write NPSV for $\Sigma_1^P SV$. For any $k \in \mathbb{N}$ and any $A, B \subseteq \Sigma^*$, a single-valued partial function f from Σ^* to Σ^* is called a Σ_k^P -*m-quasireduction* (Δ_k^P -*m-quasireduction*, resp.) from A to B if (i) f is in $\Sigma_k^P SV$ ($F\Delta_k^P$, resp.), (ii) $\text{dom}(f)$ is infinite, and (iii) for any string $x \in \text{dom}(f)$, $x \in A$ iff $f(x) \in B$. For any string $u \in \Sigma^*$, the *inverse image* $f^{-1}(u)$ of f at u is the set $\{x \in \text{dom}(f) \mid f(x) = u\}$. Notice that $f^{-1}(u) = \emptyset$ if $u \notin \text{ran}(f)$.

LEMMA 2 *Let $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$ and $S \subseteq \Sigma^*$.*

- 1 *S is \mathcal{C} -immune if and only if (i) S is infinite and (ii) for every set B , every \mathcal{C} - m -quasireduction f from S to B , and every u in B , $f^{-1}(u)$ is finite.*
- 2 *S is \mathcal{C} -bi-immune if and only if (i) S is infinite and (ii) for every set B , every \mathcal{C} - m -quasireduction f from S to B , and every u in Σ^* , $f^{-1}(u)$ is finite.*

The characterization given in Lemma 2(2) led Balcázar and Schöning [2] to introduce a stronger notion of P-bi-immunity: strong P-bi-immunity. A more general notion, called strong \mathcal{C} -immunity, will be introduced in Section 2.

Whether an NP-simple set exists is one of the long-standing open problems because such a set separates NP from co-NP. Nonetheless, NP-simple sets are known to exist in various relativized worlds. In early 1980s, Homer and Maass [8] and Balcázar [1] constructed relativized worlds where an NP-simple set exists. Later, Vereshchagin [24] proved that, relative to a random oracle, an NP-simple set exists with probability 1. From Theorem 9 in Section 2, for instance, it immediately follows that an NP-simple set exists relative to a generic oracle. Torenvliet [21] built an oracle relative to which a Σ_2^P -simple set exists. For a much higher level k of the polynomial-time

⁴A partial function f from Σ^* to Σ^* is *polynomially bounded* if there exists a polynomial p such that $|f(x)| \leq p(|x|)$ for any string $x \in \text{dom}(f)$.

hierarchy, Bruschi [4] constructed an oracle relative to which Σ_k^P -simple sets exist using the size lower bounds of certain nonuniform constant-depth circuits.

In the rest of this section, we focus on closure properties of the class of all Σ_k^P -immune sets because no such closure property has been systematically studied in the literature. A complexity class \mathcal{C} is said to be *closed downward under a reduction \leq_r on infinite sets* if, for any pair of infinite sets A and B , $A \leq_r B$ and $B \in \mathcal{C}$ imply $A \in \mathcal{C}$. Here, we study three reducibilities. Let $k \in \mathbb{N}$. A set A is Δ_k^P -*d-reducible* to B via f if f is a total $F\Delta_k^P$ -function from Σ^* to $\Sigma^{<\omega}$ (which is identified with Σ^*) and $A = \{x \mid B \cap \text{set}(f(x)) \neq \emptyset\}$, where $\text{set}(\langle y_1, y_2, \dots, y_m \rangle) = \{y_1, y_2, \dots, y_m\}$. By contrast, A is Δ_k^P -*c-reducible* to B via f if \bar{A} is Δ_k^P -*d-reducible* to \bar{B} via f . For any fixed $i \in \mathbb{N}^+$, A is Δ_k^P -*itt-reducible* to B via (f, α) if A is Δ_k^P -*tt-reducible* to B via (ν, f, α) , where $\nu(x) = 0^i$ for any x . For any reducibility r using computation \mathcal{C} , we say that A is $h\text{-}\mathcal{C}$ -*r-reducible* to B if A is \mathcal{C} -*r-reducible* to B via f (or (f, α)) such that f is componentwise honest.

Now, we claim that the class of all Σ_k^P -immune sets is closed downward under $h\text{-}\Delta_k^P$ -*c-reductions* on infinite sets; however, we cannot replace this conjunctive reducibility by disjunctive reducibility.

THEOREM 3 *Let $k \in \mathbb{N}^+$.*

- 1 *The class of all Σ_k^P -immune sets is closed downward under $h\text{-}\Delta_k^P$ -c-reductions on infinite sets.*
- 2 *The class of all NP-immune sets is not closed under $h\text{-P}$ -d-reductions or $h\text{-P}$ -2tt-reductions on infinite sets.*

The first claim of Theorem 3 is easy and is shown as follows. Assume that an infinite set A is $h\text{-}\Delta_k^P$ -*c-reducible* to a Σ_k^P -immune set B via a componentwise-honest reduction f . If A contains an infinite Σ_k^P -subset C , then consider the set $D = \{y \mid \exists x \in C[|x| \leq p(|y|) \wedge y \in \text{set}(f(x))]\}$, where p is a polynomial such that $|x| \leq p(|y|)$ for all x and all $y \in \text{set}(f(x))$. Clearly, D is an infinite Σ_k^P -subset of B , a contradiction. Therefore, A is Σ_k^P -immune.

How complex are Σ_k^P -simple sets? Intuitively, \mathcal{C} -simple sets are “thin” and thus cannot be “complete” for the class \mathcal{C} . As an immediate consequence of Theorem 3(1), we obtain the following corollary.

COROLLARY 4 *Let $k \in \mathbb{N}^+$. No Σ_k^P -simple set is $h\text{-}\Delta_k^P$ -d-complete for Σ_k^P .*

Recently, Agrawal (cited in [19]) showed, using the NP-levelability of $\overline{\text{SAT}}$ (assuming $\text{SAT} \not\subseteq \text{P}$), that no NP-simple set is $h\text{-P}$ -btt-complete for NP, where SAT is the set of all satisfiable quantifier-free Boolean formulas. His argument will be generalized in Section 4 in connection to \mathcal{C} -hyperimmune sets.

2. Strong Immunity and Strong Simplicity

Following the introduction of P-bi-immunity, Balcázar and Schöning [2] stepped forward to introduce the notion of strongly P-bi-immunity, which comes from the quasireducibility-characterization of P-bi-immunity given in Lemma 2(2). While P-bi-immunity requires its quasireductions to be finite-to-one, strong P-bi-immunity requires the quasireductions to be almost one-to-one, where a quasireduction f is called

almost one-to-one on a set S if the collision set $\{(x, y) \in (\text{dom}(f) \cap S)^2 \mid x < y \wedge f(x) = f(y)\}$ is finite. Such strongly P-bi-immune sets are known to exist even in the class E [2].

Generalizing the notion of P-bi-immunity, we can introduce strong \mathcal{C} -bi-immunity for any complexity class \mathcal{C} lying in the polynomial-time hierarchy. Moreover, we newly introduce the notions of strong \mathcal{C} -immunity and strong \mathcal{C} -simplicity. Recall that Σ_k^P -m-quasireductions are all single-valued functions in $\Sigma_k^P\text{SV}$ for each $k \in \mathbb{N}^+$.

DEFINITION 5 *Let $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$.*

- 1 *A set S is strongly \mathcal{C} -immune if (i) S is infinite and (ii) for every set B and every \mathcal{C} -m-quasireduction f from S to B , f is almost one-to-one on S .*
- 2 *A set S is strongly \mathcal{C} -bi-immune if S and \bar{S} are both strongly \mathcal{C} -immune.*
- 3 *A set S is strongly \mathcal{C} -simple if S is in \mathcal{C} and \bar{S} is strongly \mathcal{C} -immune.*

In particular, when $\mathcal{C} = \text{P}$, Definition 5(2) coincides with the notion of P-bi-immunity originally given in [2].

LEMMA 6 *For any complexity class $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$, every strongly \mathcal{C} -immune set is \mathcal{C} -immune and every strongly \mathcal{C} -simple set is \mathcal{C} -simple.*

A major difference between \mathcal{C} -immunity and strong \mathcal{C} -immunity is shown in the following example. For any NP-immune set A , the disjoint union⁵ $A \oplus A$ is also NP-immune; on the contrary, it is not strongly NP-immune because $A \oplus A$ can be reduced to A by the almost two-to-one function f defined as $f(\lambda) = \lambda$ and $f(xb) = x$ for $b \in \{0, 1\}$, where λ is the empty string. Therefore, the class of all strongly NP-immune sets is not closed under the disjoint-union operator. Historically, using the structural difference between these two notions, Balcázar and Schöning [2] constructed a set in E which is P-bi-immune but not strongly P-bi-immune.

We show a closure property of the class of strongly Σ_k^P -immune sets. If A is Δ_k^P -m-reducible to B via a one-to-one honest reduction f , we say that A is *$h\text{-}\Delta_k^P\text{-1-reducible}$* to B via f .

PROPOSITION 7 *Let $k \in \mathbb{N}^+$.*

- 1 *The class of all strongly Σ_k^P -immune sets is closed downward under $h\text{-}\Delta_k^P\text{-1-reductions}$ on infinite sets.*
- 2 *The class of all strongly NP-immune sets is not closed downward under $h\text{-P-m-reductions}$ on infinite sets.*

COROLLARY 8 *For each level $k \in \mathbb{N}^+$, there is no strongly Σ_k^P -simple set that is $h\text{-}\Delta_k^P\text{-1-complete}$ for Σ_k^P .*

Finally, we turn our interest to relativization. For each $k \in \mathbb{N}^+$, it is easy to show that a strongly Σ_k^P -simple set exists relative to a recursive oracle (similar to Proposition

⁵The disjoint union of A and B is the set $\{0x \mid x \in A\} \cup \{1x \mid x \in B\}$.

26). Even relative to a random oracle, there exists a strongly NP-simple set with probability 1 (similar to Proposition 27). Employing weak forcing, we now prove the following relativization result.

THEOREM 9 *A strongly NP^G -simple set exists relative to a generic oracle G .*

3. Almost Immunity and Almost Simplicity

We have shown in the previous section that strong \mathcal{C} -immunity and its simplicity strengthen the ordinary notion of \mathcal{C} -immunity and \mathcal{C} -simplicity. In contrast to these notions, Orponen [14] and Orponen, Russo, and Schöning [16] expanded P-immunity to the new notion of almost P-immunity. The complementary notion of almost P-immunity under the term P-levelability (a more general term “levelable sets” was first used by Ko [10] in a resource-bounded setting) was extensively discussed by Orponen et al. [16]. Naturally, we can generalize these notions to almost \mathcal{C} -immunity and \mathcal{C} -levelability for any complexity class \mathcal{C} . Furthermore, we newly introduce the notion of almost \mathcal{C} -bi-immunity and almost \mathcal{C} -simplicity.

DEFINITION 10 *Let \mathcal{C} be any complexity class.*

- 1 *A set S is almost \mathcal{C} -immune if S is the union of a \mathcal{C} -immune set and a set in \mathcal{C} .*
- 2 *An infinite set is \mathcal{C} -levelable if it is not almost \mathcal{C} -immune.*
- 3 *A set S is almost \mathcal{C} -bi-immune if S and \bar{S} are both almost \mathcal{C} -immune.*
- 4 *A set S is almost \mathcal{C} -simple if S is an infinite set in \mathcal{C} and \bar{S} is the union of a set A in \mathcal{C} and a \mathcal{C} -immune set B , where the difference $B \setminus A$ is infinite.*

It follows from Definition 10(1) that every almost \mathcal{C} -immune set is infinite since so is every \mathcal{C} -immune set. The definition of almost \mathcal{C} -simplicity in Definition 10(4) is slightly different from other simplicity definitions because the infinity condition of the difference $B \setminus A$ is necessary to guarantee $\mathcal{C} \neq \text{co-}\mathcal{C}$, provided that an almost \mathcal{C} -simple set exists.

LEMMA 11 *Let \mathcal{C} be any complexity class closed under finite variations, finite union and finite intersection. If an almost \mathcal{C} -simple set exists, then $\mathcal{C} \neq \text{co-}\mathcal{C}$.*

Lemma 11 is shown as follows. Take any set S such that $\bar{S} = A \cup B$ for a set $A \in \mathcal{C}$ and a \mathcal{C} -immune set B . Suppose $\mathcal{C} = \text{co-}\mathcal{C}$. Note that $B \setminus A \subseteq B$ and $B \setminus A = \bar{S} \setminus A \in \mathcal{C}$. Since B is \mathcal{C} -immune, $B \setminus A$ must be finite.

The following lemma is immediate from Definition 10.

LEMMA 12 *For any complexity class \mathcal{C} , every \mathcal{C} -immune set is almost \mathcal{C} -immune and every \mathcal{C} -simple set is almost \mathcal{C} -simple.*

Several characterizations of almost P-immunity and P-levelability are shown in [16] in terms of maximal P-subsets and P-to-finite reductions. We can naturally expand these characterizations to almost Δ_k^P -immunity and Δ_k^P -levelability (but not to the Σ -level classes of the polynomial-time hierarchy).

To understand the characteristics of almost \mathcal{C} -immunity, we begin with a simple observation. It is known in [16] that any honestly paddable⁶ set not in P is P -levelable. As observed in [18], the essence of this assertion is that if $A \notin P$ and A is length-increasing P - m -selfreducible then A is P -levelable, where A is *length-increasing \mathcal{C} - m -selfreducible* if A is \mathcal{C} - m -reducible to A via a certain length-increasing reduction. This observation can be generalized to Δ_k^P -levelable sets in the following lemma.

LEMMA 13 *Let $k \in \mathbb{N}^+$ and $A \subseteq \Sigma^*$. Assuming that $A \notin \Delta_k^P$, if A is length-increasing Δ_k^P - m -selfreducible, then A and \bar{A} are both Δ_k^P -levelable. Thus, if $\Delta_k^P \neq \Sigma_k^P$ then Σ_k^P as well as Π_k^P has a Δ_k^P -levelable set.*

Most known NP - m -complete sets are known to be honestly paddable and thus, by Lemma 13, the complements of these sets are P -levelable sets, which are also NP -levelable. Therefore, most known NP - m -complete sets cannot be almost NP -simple. This result can be compared to Proposition 16.

Now, we assume a standard effective enumeration $\{\varphi_s\}_{s \in \Sigma^*}$ of all nondeterministic TMs φ_s . For each index s , define the set $W_s = \{x \mid \varphi_s(x) \downarrow = 1\}$, where “ $\varphi_s(x) \downarrow$ ” means that φ_s eventually halts on input x . Fix $k \in \mathbb{N}$. Let $NP_{(k)}$ denote the collection of all sets W_s such that, for any string $x \in W_s$, the running time of φ_s on input x is at most $|s| \cdot |x|^k + |s|$. Moreover, we set $INDEX_{(k)} = \{s \mid W_s \in NP_{(k)}\}$. Note that $NP = \bigcup_{k \in \mathbb{N}} NP_{(k)}$.

Earlier, Ko and Moore [11] considered the resource-bounded notion of “productive sets.” Another formulation based on $NP_{(k)}$ was later given by Joseph and Young [9], who used the terminology of k -creative⁷ sets, where k is any number in \mathbb{N}^+ . They showed that every k -creative set is P - m -complete for NP . Orponen et al. [16] showed that, unless $P = NP$, every honestly k -creative set is P -levelable by demonstrating that any honestly k -creative set is length-increasing P - m -selfreducible. From Lemma 13, it follows that any honestly k -creative set and its complement are both P -levelable. Consequently, we obtain the following result.

COROLLARY 14 *For any $k \in \mathbb{N}^+$, no honestly k -creative set is almost NP -simple.*

Our notion of almost \mathcal{C} -simplicity is similar to what Uspenskii [23] discussed under the term “pseudosimplicity.” Here, we give a resource-bounded version of his pseudosimplicity. A set S is called *\mathcal{C} -pseudosimple* if there is an infinite \mathcal{C} -subset A of \bar{S} such that $S \cup A$ is \mathcal{C} -simple. Although \mathcal{C} -simple sets cannot be \mathcal{C} -pseudosimple by our definition, any infinite \mathcal{C} -pseudosimple set is almost \mathcal{C} -simple. The latter claim is shown as follows. Suppose that S is an infinite \mathcal{C} -pseudosimple set and A is a \mathcal{C} -subset of \bar{S} for which $S \cup A$ is \mathcal{C} -simple. This means that $\bar{S} \setminus A$ is \mathcal{C} -immune. Therefore, S is almost \mathcal{C} -simple.

⁶A set S is *polynomially paddable* (paddable, in short) if there is an one-to-one total FP-function pad (called the *padding function*) from Σ^* to Σ^* such that, for all pairs $(x, y) \in \Sigma^* \times \Sigma^*$, $x \in S$ iff $pad((x, y)) \in S$. A set S is *honestly paddable* if it is paddable with a padding function that is component-wise honest.

⁷A set S is called *k -creative* if there exists a function $f \in FP$ such that, for any index $i \in INDEX_{(k)}$, $f(i) \in S$ iff $f(i) \in W_i$. This function f is called the *productive function* for S . If in addition f is honest, S is called *honestly k -creative*.

The following theorem shows a close connection among simplicity, almost simplicity, and pseudosimplicity.

THEOREM 15 *For each $k \in \mathbb{N}^+$, the following three statements are equivalent.*

- 1 *There exists a Σ_k^P -simple set.*
- 2 *There exists an infinite Σ_k^P -pseudosimple set in P .*
- 3 *There exists an almost Σ_k^P -simple set in P .*

The most essential part of Theorem 15 is the implication from 1 to 2. Assume that S is an almost Σ_k^P -simple set. Let $A_1 = 1S$ and $B_1 = 1\bar{S}$. Clearly, B_1 is infinite. Note that A_1 is a Σ_k^P -subset of $1\Sigma^*$. Since \bar{S} is Σ_k^P -immune and $B_1 \subseteq 1\bar{S}$, B_1 is Σ_k^P -immune. Since $B_1 = 1\Sigma^* \cap \overline{A_1}$, the set $0\Sigma^* \cup A_1$ is Σ_k^P -simple. Hence, $0\Sigma^*$ is Σ_k^P -pseudosimple, as required. Similarly, $1\Sigma^*$ is Σ_k^P -pseudosimple.

Theorem 15 indicates the importance of the structure of P in the course of the study of Σ_k^P -simplicity. In a relativized world where a Σ_k^P -simple set exists [4], since Theorem 15 relativizes, there exists an almost Σ_k^P -simple set within P .

Finally, we briefly discuss a closure property of the class of all almost Σ_k^P -immune sets under polynomial-time reductions. For each number $k \in \mathbb{N}^+$, the class of all almost Σ_k^P -immune sets is closed under $h\text{-}\Delta_k^P$ -m-reductions on infinite sets. This immediately implies the following consequence.

PROPOSITION 16 *For each number $k \in \mathbb{N}^+$, there is no almost Σ_k^P -simple set that is $h\text{-}\Delta_k^P$ -m-complete for Σ_k^P .*

4. Hyperimmunity and Hypersimplicity

Since Post [17] constructed a so-called *hypersimple* set, the notions of hyperimmunity and hypersimplicity have played a significant role in the progress of classical recursion theory. A resource-bounded version of these notions was first considered by Yamakami [26] and studied extensively by Schaefer and Fenner [19]. The definition of Schaefer and Fenner is based on the notion of “honest NP-arrays”, which differs from the notion of “strong arrays” in recursion theory, where a strong array is a series of pairwise disjoint finite sets. For our formalization, we demand only “eventually disjointness” for sets in an array rather than “pairwise disjointness.”

A binary string x is said to *represent* a finite set $\{a_1, a_2, \dots, a_k\}$ if and only if $x = \langle a_1, a_2, \dots, a_k \rangle$ and $a_1 < a_2 < \dots < a_k$ in the lexicographic order on Σ^* . For convenience, we say that a set A *surpasses* another set B if there exists a string $z \in A$ satisfying $z > x$ (lexicographically) for all strings x in B .

DEFINITION 17 *Let $k \in \mathbb{N}^+$, $A \subseteq \Sigma^*$, and $C \in \{\Sigma_k^P, \Delta_k^P\}$.*

- 1 *An infinite sequence $\mathcal{D} = \{D_s\}_{s \in \Sigma^*}$ of finite sets is called a Σ_k^P -array (Δ_k^P -array, resp.) if there exists a single-valued partial function f in $\Sigma_k^P\text{SV}$ ($F\Delta_k^P$, resp.) such that (i) $\text{dom}(f)$ is infinite, (ii) $D_s \neq \emptyset$ and $f(s)$ represents D_s for any string $s \in \text{dom}(f)$, and (iii) $D_s = \emptyset$ for any string $s \notin \text{dom}(f)$. This f is called the supporting function of \mathcal{D} and the set $\bigcup_{s \in \text{dom}(f)} D_s$ is called the support of \mathcal{D} . The width of \mathcal{D} is the supremum of the cardinality $|D_s|$ over all $s \in \text{dom}(f)$.*

- 2 A \mathcal{C} -array \mathcal{D} has an infinite support if the support of \mathcal{D} is infinite.
- 3 A \mathcal{C} -array $\{D_s\}_{s \in \Sigma^*}$ via f is polynomially honest (honest, in short) if f is componentwise honest; namely, there exists a polynomial p satisfying that $|s| \leq p(|x|)$ for any $s \in \text{dom}(f)$ and any $x \in D_s$.
- 4 A \mathcal{C} -array $\{D_s\}_{s \in \Sigma^*}$ via f is eventually disjoint if, for every string x in $\text{dom}(f)$, there exists a string y in $\text{dom}(f)$ such that $y \geq x$ (lexicographically), D_y surpasses D_x , and $D_x \cap D_y = \emptyset$.
- 5 A \mathcal{C} -array $\{D_s\}_{s \in \Sigma^*}$ via f intersects A if $D_s \cap A \neq \emptyset$ for all $s \in \text{dom}(f)$.

The honesty condition of an \mathcal{C} -array guarantees that the array is eventually disjoint. In addition, any eventually-disjoint \mathcal{C} -array has an infinite support because, for any element D in the array, we can always find another disjoint element D' .

A simple relationship between Σ_k^{P} -simplicity and a honest Σ_k^{P} -array is given in the following lemma, which was implicitly proven by Yamakami [26] and later explicitly stated in [19] for the case where $k = 1$.

LEMMA 18 *Let $k \in \mathbb{N}^+$ and let A be any Σ_k^{P} -simple set. For every number $\ell \in \mathbb{N}^+$, there is no honest Σ_k^{P} -array \mathcal{D} such that (i) the width of \mathcal{D} is at most ℓ and (ii) \mathcal{D} intersects A .*

We introduce below the notions of \mathcal{C} -hyperimmunity and honest \mathcal{C} -hyperimmunity.

DEFINITION 19 *Let $\mathcal{C} \in \{\Delta_k^{\text{P}}, \Sigma_k^{\text{P}} \mid k \in \mathbb{N}\}$.*

- 1 A set S is (honestly) \mathcal{C} -hyperimmune if S is infinite and there is no (honest) \mathcal{C} -array \mathcal{D} such that \mathcal{D} is eventually disjoint and \mathcal{D} intersects A .
- 2 A set S is (honestly) \mathcal{C} -bi-hyperimmune if S and \bar{S} are both (honestly) \mathcal{C} -hyperimmune.
- 3 A set S is (honestly) \mathcal{C} -hypersimple if $S \in \mathcal{C}$ and \bar{S} is (honestly) \mathcal{C} -hyperimmune.

Note that “NP-hyperimmunity” defined by Schaefer and Fenner [19] coincides with our honest NP-hyperimmunity. The following relationship between immunity and hyperimmunity can be obtained immediately from Definition 19.

LEMMA 20 *For any complexity class $\mathcal{C} \in \{\Sigma_k^{\text{P}}, \Delta_k^{\text{P}} \mid k \in \mathbb{N}\}$, every honestly \mathcal{C} -hyperimmune set is \mathcal{C} -immune and every honestly \mathcal{C} -hypersimple set is \mathcal{C} -simple.*

In late 1970s, Selman [20] introduced the notion of P-selective sets, which are analogues of semi-recursive sets in recursion theory. These sets connects P-immunity to P-hyperimmunity. In general, for any class \mathcal{F} of total functions, we say that a set S is \mathcal{F} -selective if there exists a function (called the selector) f in \mathcal{F} such that, for all pairs $(x, y) \in \Sigma^* \times \Sigma^*$, (i) $f(x, y) \in \{x, y\}$ and (ii) $\{x, y\} \cap S \neq \emptyset$ implies $f(x, y) \in S$. Recall the partial function class $\Sigma_k^{\text{P}}\text{SV}$. We use the notation $\Sigma_k^{\text{P}}\text{SV}_t$ to denote the collection of all total functions in $\Sigma_k^{\text{P}}\text{SV}$.

LEMMA 21 *Let $k \in \mathbb{N}^+$. Every Σ_k^{P} -immune $\Sigma_k^{\text{P}}\text{SV}_t$ -selective set is honestly Σ_k^{P} -hyperimmune.*

We give the proof of Lemma 21. Assume that S is $\Sigma_k^P \text{SV}_t$ -selective via a selector f and has a honest Σ_k^P -array $\mathcal{D} = \{D_s\}_{s \in \Sigma^*}$ intersecting S via a supporting function g . For each $y \in \text{dom}(g)$, assuming $D_y = \{x_1, x_2, \dots, x_m\}$ with $x_1 < x_2 < \dots < x_m$, let $y_1 = x_1$ and $y_{i+1} = f(y_i, x_{i+1})$ for every $i \in \{1, 2, \dots, m-1\}$ and then define $h(y) = y_m$. This h is honest and in $\Sigma_k^P \text{SV}$. The set $B = \{x \mid \exists y[y \in \text{dom}(h) \wedge h(y) = x]\}$ is therefore infinite and in Σ_k^P since $\text{dom}(f)$ is infinite and h is honest. Because $B \subseteq S$, B cannot be Σ_k^P -immune. Note that our proof relativizes.

Observe that the complement of a $\Sigma_k^P \text{SV}_t$ -selective set S is also $\Sigma_k^P \text{SV}_t$ -selective because the exchange of the output string of any selector for S gives rise to a selector for \bar{S} . It also follows from Lemma 21 that every NP-simple P-selective set is honestly NP-hypersimple since the complement of any P-selective set is also P-selective.

Next, we show that strong P-immunity does not imply honest P-hyperimmunity within the class E. Earlier, Balcázar and Schöning [2] created a strongly P-bi-immune set S in E with the density $|S \cap \Sigma^{\leq n}| = 2^{n+1} - n - 1$ for all $n \in \mathbb{N}$. For each x , let D_x consist of the first $|x| + 1$ elements of $\Sigma^{|x|}$. Clearly, D_x intersects S . This implies that S is not honestly P-hyperimmune. Therefore, we obtain the following proposition.

PROPOSITION 22 *There exists a strongly P-bi-immune set in E that is not honestly P-hyperimmune.*

As a main theorem, we show the P-T-incompleteness of Σ_k^P -hypersimple sets. Generally, we say that A is Δ_k^P -T-reducible to B if there exists an oracle Δ_k^P -machine M which recognizes A with access to B as an oracle. If in addition M on input x makes only queries y to B that satisfy $|x| \leq p(|y|)$, where p is a fixed polynomial, then we say that A is $h\text{-}\Delta_k^P$ -T-reducible to B via M . This reduction machine M is simply called *honest*.

THEOREM 23 *Let $k \in \mathbb{N}^+$.*

- 1 *No Σ_k^P -hypersimple set is P-T-complete for Σ_k^P .*
- 2 *No honestly Σ_k^P -hypersimple set is h-P-T-complete for Σ_k^P .*

Note that it is not clear if we can replace the P-T-completeness in Theorem 23 by the Δ_k^P -T-completeness. Theorem 23 follows from Lemma 24 in the following fashion. We prove only the first claim. Assume that B is a Σ_k^P -hypersimple set that is P-T-complete for Σ_k^P . Thus, $\Delta_k^P \neq \Sigma_k^P$. Clearly, B is in EXP and every Σ_k^P -set is P-T-reducible to B . By Lemma 24, every Σ_k^P -set is almost Δ_k^P -immune. This contradicts Lemma 13. Therefore, B cannot be Σ_k^P -hypersimple.

LEMMA 24 *Let $k \in \mathbb{N}^+$ and let A be any infinite set in Σ_k^P .*

- 1 *If A is P-T-reducible to a Σ_k^P -hyperimmune set in EXP, then A is almost Δ_k^P -immune.*
- 2 *If A is h-P-T-reducible to a honestly Σ_k^P -hyperimmune set, then A is almost Δ_k^P -immune.*

Lemma 24 needs a key idea of Agrawal (mentioned earlier), who showed that no NP-simple set is h-P-btt-complete for NP. We extend his core argument to Lemma

25. For convenience, a complexity class \mathcal{C} is said to be *closed under intersection with Δ_k^P -sets* if, for any set A in \mathcal{C} and any set B in Δ_k^P , the intersection $A \cap B$ is in \mathcal{C} .

LEMMA 25 *Let \mathcal{C} be any complexity class containing Δ_k^P such that \mathcal{C} is closed under intersection with Δ_k^P -sets. Let A be any Δ_k^P -levelable set in \mathcal{C} . If A is Δ_k^P -T-reducible to B via a reduction machine M , then there exists an infinite set C in \mathcal{C} such that $Q(M, B, x) \cap B \neq \emptyset$ for all $x \in C$.*

Bruschi [4] demonstrated how to construct a recursive oracle relative to which a Σ_k^P -simple set exists. We can easily modify his proof to obtain a P-selective set that is Σ_k^P -simple in a relativized world. Since Lemma 21 relativizes, we obtain the following proposition.

PROPOSITION 26 *For each $k \in \mathbb{N}^+$, there exists a recursive oracle A such that a $\Sigma_k^P(A)$ -hypersimple set exists.*

As Schaefer and Fenner [19] demonstrated, it is relatively easy to prove the existence of an honest NP^G -hypersimple set relative to a generic oracle G . By contrast, Vereshchagin [24] proved the existence of an NP-simple set relative to a random oracle with probability 1. Again, we modify his proof to construct a relativized P-selective NP-simple set. From Lemma 21, the next proposition follows.

PROPOSITION 27 *With probability 1, an honestly NP^X -hypersimple set exists relative to a random oracle X .*

An important open problem is to prove that, at each level k of the polynomial-time hierarchy, honest Σ_k^P -hypersimple sets exist relative to a random oracle with probability 1.

5. Completeness Under Non-Honest Reductions

Immunity has a deep connection to various completeness notions. For example, there is a simple, tt-complete set; however, no simple set is btt-complete. In the previous sections, we have shown that various types of resource-bounded simple sets cannot be complete under certain polynomial-time honest reductions. This section instead focuses on the incompleteness of simple sets under non-honest reductions.

To remove the honesty condition from reductions, we often need to make extra assumptions for similar incompleteness results. In mid 1980s, Hartmanis, Li, and Yesha [6] proved that (i) no NP-immune set in EXP is P-m-hard for NP if $\text{NP} \not\subseteq \text{SUBEXP}$ and (ii) no NP-simple set is P-m-complete if $\text{NP} \cap \text{co-NP} \not\subseteq \text{SUBEXP}$. These results can be expanded to any Δ -level of the polynomial-time hierarchy and of the subexponential-time hierarchy⁸. We also improve the latter claim.

⁸The Δ -level of the subexponential-time hierarchy is defined as: $\text{SUB}\Delta_0^{\text{EXP}} = \text{SUBEXP}$ and $\text{SUB}\Delta_{k+1}^{\text{EXP}} = \text{SUBEXP}^{\Sigma_k^P}$ for every $k \in \mathbb{N}$, where SUBEXP^A denotes $\bigcap_{\epsilon > 0} \text{DTIME}^A(2^{n^\epsilon})$ for any oracle A . When $A = \emptyset$, we simply write SUBEXP for SUBEXP^A .

To describe our expansion, we need the unambiguous complexity class $U(\Sigma_k^P \cap \Pi_k^P)$ introduced by Yamakami [25]. For any complexity class \mathcal{C} , a set A is in $U(\mathcal{C})$ (or UC) if there exists a single-valued partial function f such that (i) f is polynomially bounded, (ii) $Graph(f) \in \mathcal{C}$, and (iii) $A = \text{dom}(f)$ [25].

PROPOSITION 28 *Let j and k be any nonnegative integers.*

- 1 *No Σ_k^P -immune set in Δ_j^{EXP} is Δ_k^P - m -hard for Σ_k^P if $\Sigma_k^P \not\subseteq \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$.*
- 2 *No Σ_k^P -simple set is Δ_k^P - m -complete for Σ_k^P if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_k^{\text{EXP}}$.*

Note that Proposition 28(2) directly follows from Theorem 29(2).

The original result of Hartmanis et al. refers to the P- m -incompleteness of NP-simple sets. Recently, Schaefer and Fenner [19] showed a similar result for the P-1tt-completeness. They proved that no NP-simple set is P-1tt-complete for NP if $UP \not\subseteq \text{SUBEXP}$. A key to their proof is the fact⁹ that $Sep(\text{SUBEXP}, \text{NP})$ implies $UP \subseteq \text{SUBEXP}$, where $Sep(\mathcal{C}, \mathcal{D})$ means the separation property in [25] that, for any two disjoint sets $A, B \in \mathcal{D}$, there exists a set $S \in \mathcal{C} \cap \text{co-}\mathcal{C}$ satisfying that $A \subseteq S \subseteq \bar{B}$.

The following theorem shows that the assumption $UP \not\subseteq \text{SUBEXP}$ in [19] can be replaced by $U(\text{NP} \cap \text{co-NP}) \not\subseteq \text{SUBEXP}$.

THEOREM 29 *Let $j, k \in \mathbb{N}^+$.*

- 1 *No Σ_k^P -immune set in Δ_j^{EXP} is Δ_k^P -1tt-hard for $U(\Sigma_k^P \cap \Pi_k^P)$ if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$.*
- 2 *No Σ_k^P -simple set is Δ_k^P -1tt-complete for Σ_k^P if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_k^{\text{EXP}}$.*

Theorem 29 follows from the technical lemmas: Lemmas 30 and 31. The proof for its second claim proceeds as follows. Assume that B is Δ_k^P -1tt-complete for Σ_k^P . Choose an infinite set $A \in U(\Sigma_k^P \cap \Pi_k^P) - \text{SUB}\Delta_k^{\text{EXP}}$, which is of the form $\{x \mid \exists y[\langle x, y \rangle \in Graph(f)]\}$ for a certain polynomially-bounded partial function f whose graph is in $\Sigma_k^P \cap \Pi_k^P$. Similar to [25], set $A_1 = \{\langle x, z \rangle \mid \exists y[z \leq y \wedge \langle x, y \rangle \in Graph(f)]\}$ and $A_2 = \{\langle x, z \rangle \mid \exists y[z < y \wedge \langle x, y \rangle \in Graph(f)]\}$. Clearly, A_1 and A_2 are in $U(\Sigma_k^P \cap \Pi_k^P) - \text{SUB}\Delta_k^{\text{EXP}}$. Since A_1 is Δ_k^P -1tt-reducible to B , by Lemma 30, there exists a set $C \in \Delta_k^P$ such that $A_1 \cap C$ is infinite and coinfinite and $A_1 \cap C$ is $h\text{-}\Delta_k^P$ -1tt-reducible to B . Applying Lemma 31(1), we obtain a set $D \in \Delta_k^P$ and a total $F\Delta_k^P$ -function f such that $\overline{A_1 \cap C} \cap \overline{D}$ is finite, f Δ_k^P - m -reduces $A_1 \cap C$ to \overline{B} , and f is honest on the domain D . Since $A_2 \subseteq \overline{A_1 \cap C}$, $f(A_2) \subseteq \overline{B}$. Moreover, $A_2 \cap D$ is infinite. The honesty of f on D implies that \overline{B} has an infinite Σ_k^P -subset $f(A_2 \cap D)$.

The key idea of Hartmanis et al. [6] is to find a set that can be honestly reducible. Lemma 30 is a “1tt” version of a technical part of [6].

LEMMA 30 *Let $j, k \in \mathbb{N}^+$. Assume that $A \notin \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$ and $B \in \Delta_j^{\text{EXP}}$. If A is Δ_k^P -1tt-reducible to B , then there exists a set C in Δ_k^P such that (i) $A \cap C$ is $h\text{-}\Delta_k^P$ -1tt-reducible to B and (ii) $A \cap C$ and $\overline{A} \cap C$ are infinite and coinfinite.*

⁹Actually, the result of Schaefer and Fenner can be strengthened in the following way: $Sep(\text{SUBEXP}, U(\text{NP} \cap \text{co-NP}))$ if and only if $U(\text{NP} \cap \text{co-NP}) \subseteq \text{SUBEXP}$. This is obtained by analyzing a similar result in [25].

LEMMA 31 *Let $k \in \mathbb{N}^+$ and $A, B \subseteq \Sigma^*$. Assume that B is Σ_k^P -immune and A is $h\text{-}\Delta_k^P$ -1tt-reducible to B .*

1 *If $A \in \Sigma_k^P$, then there exist a set $D \in \Delta_k^P$ and a total function $f \in \text{F}\Delta_k^P$ such that $\overline{A} \cap \overline{D}$ is finite, f Δ_k^P -m-reduces A to \overline{B} , and f is honest on the domain D .*

2 *A belongs to $\Sigma_k^P \cap \Pi_k^P$ if and only if A belongs to Δ_k^P .*

6. Limited Immunity and Simplicity

Within our current knowledge, we cannot prove or disprove the existence of an NP-simple set. The difficulty comes from the fact that an NP-immune set requires every NP-subset to be finite. If we restrict our attention to certain types of NP-subsets, then we may overcome the difficulty. Under the name of k -immune sets, Homer [7] required only $\text{NP}_{(k)}$ -subsets, for a fixed number k , to be finite. He then demonstrated how to construct a k -simple set within NP using Ladner's delayed diagonalization technique.

In this section, we investigate the notions obtained by restricting the requirements for immunity and simplicity. We first review Homer's notions of k -immunity and k -simplicity.

DEFINITION 32 *Let k be any number in \mathbb{N}^+ .*

1 *A set S is k -immune if S is infinite and there is no index i in $\text{INDEX}_{(k)}$ such that W_i is infinite and $W_i \subseteq S$.*

2 *A set S is k -simple if S belongs to NP and \overline{S} is k -immune.*

An "effective" version of immune and simple sets, called effectively immune and effectively simple sets, has been studied in recursion theory. Effectively simple sets are known to be T-complete and there also exists an effectively simple tt-complete set. If A is strongly effectively immune, then \overline{A} cannot be immune. Analogously, we consider a resource-bounded version of such effectively immune and simple sets. Here, we freely identify binary strings with natural numbers using the lexicographic order on Σ^* .

DEFINITION 33 *Let $k \in \mathbb{N}^+$.*

1 *A set S is feasibly k -immune if (i) S is infinite and (ii) there exists a polynomial p such that, for every index i in $\text{INDEX}_{(k)}$, $W_i \subseteq S$ implies $|W_i| \leq 2^{p(i)}$.*

2 *A set S is feasibly k -simple if S is in NP and \overline{S} is feasibly k -immune.*

We can easily prove the existence of a feasibly k -immune set in Δ_2^P for each $k \in \mathbb{N}^+$. From Definition 33, every feasibly k -simple set is k -simple. The converse, however, does not hold since there exists a k -simple set which is not feasibly k -simple for each number k in \mathbb{N}^+ . The theorem below is slightly stronger than this claim since any feasibly k -simple set is also feasibly 1-simple.

THEOREM 34 *For each $k \in \mathbb{N}^+$, there exists a k -simple set which is not feasibly 1-simple.*

We return to the old question of whether NP-simple sets exist. There seems no strong evidence that suggests the existence of such a set. Only relativization provides a world where NP-simple sets exist. At the same time, we can also construct another world where these sets do not exist. These relativization results clearly indicate that the question of whether NP-simple sets exist needs unrelativizable proof techniques.

In the past few decades, the Berman-Hartmanis isomorphism conjecture has served as a working hypothesis in connection to NP-complete problems. By contrast, there has been no “natural” working hypothesis that yields the existence of NP-simple sets. For example, the hypothesis $P \neq NP$ does not suffice since Homer and Maass [8] showed a relativized world where the assumption $P \neq NP$ does not imply the existence of an NP-simple set. Motivated by Homer’s k -simplicity result, we propose the following working hypothesis:

- **The k -immune hypothesis:** There exists a positive integer k such that every infinite NP set has an infinite $NP_{(k)}$ -subset.

Under this hypothesis, we can derive the desired consequence: the existence of NP-simple sets.

LEMMA 35 *If the k -immune hypothesis holds, then there exists an NP-simple set.*

Assume that the k -immune hypothesis is true; that is, there exists a positive integer k such that every infinite NP-set has an infinite $NP_{(k)}$ -subset. Consider any k -simple set A . We claim that A is NP-simple. If A is not NP-simple, then \bar{A} has an infinite NP-subset B . By our assumption, B contains an infinite $NP_{(k)}$ -subset. Hence, A cannot be k -simple, a contradiction. Therefore, A is NP-simple.

To close this section, we claim the following result concerning the k -immune hypothesis. The proof uses weak forcing.

PROPOSITION 36 *The k -immune hypothesis fails relative to a generic oracle.*

Final Note. All the proofs that are omitted from this extended abstract will appear in its forthcoming complete version.

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