

Theoretical Computer Science Cheat Sheet

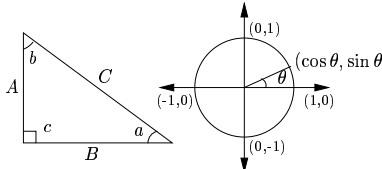
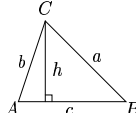
Definitions		Series	
$f(n) = O(g(n))$	iff \exists positive c, n_0 such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$.	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$.	
$f(n) = \Omega(g(n))$	iff \exists positive c, n_0 such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$.	In general: $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$	
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.	$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}$.	
$f(n) = o(g(n))$	iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$.	Geometric series: $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, c \neq 1, \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, c < 1,$	
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a < \epsilon, \forall n \geq n_0$.	$\sum_{i=0}^{\infty} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, c \neq 1, \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, c < 1.$	
sup S	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$.	Harmonic series: $H_n = \sum_{i=1}^n \frac{1}{i}, \sum_{i=1}^n iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}$.	
inf S	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$.	$\sum_{i=1}^n H_i = (n+1)H_n - n, \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right)$.	
$\liminf_{n \rightarrow \infty} a_n$	$\liminf_{n \rightarrow \infty} \{a_i \mid i \geq n, i \in \mathbb{N}\}$.		
$\limsup_{n \rightarrow \infty} a_n$	$\limsup_{n \rightarrow \infty} \{a_i \mid i \geq n, i \in \mathbb{N}\}$.		
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.		
$[n_k]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, 2. \sum_{k=0}^n \binom{n}{k} = 2^n, 3. \binom{n}{k} = \binom{n}{n-k},$	
$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.	4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$	
$\langle n_k \rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$	
$\langle\langle n_k \rangle\rangle$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$	
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, 11. \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1,$	
		12. $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1, 13. \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\},$	
14. $\left[\begin{matrix} n \\ 1 \end{matrix} \right] = (n-1)!,$	15. $\left[\begin{matrix} n \\ 2 \end{matrix} \right] = (n-1)!H_{n-1},$	16. $\left[\begin{matrix} n \\ n \end{matrix} \right] = 1,$	17. $\left[\begin{matrix} n \\ k \end{matrix} \right] \geq \left\{ \begin{matrix} n \\ k \end{matrix} \right\},$
18. $\left[\begin{matrix} n \\ k \end{matrix} \right] = (n-1) \left[\begin{matrix} n-1 \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right],$	19. $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \left[\begin{matrix} n \\ n-1 \end{matrix} \right] = \binom{n}{2},$	20. $\sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right] = n!,$	21. $C_n = \frac{1}{n+1} \binom{2n}{n},$
22. $\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ n-1 \end{matrix} \right\rangle = 1,$	23. $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ n-1-k \end{matrix} \right\rangle,$	24. $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle,$	
25. $\left\langle \begin{matrix} 0 \\ k \end{matrix} \right\rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\left\langle \begin{matrix} n \\ 1 \end{matrix} \right\rangle = 2^n - n - 1,$	27. $\left\langle \begin{matrix} n \\ 2 \end{matrix} \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$	
28. $x^n = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{n},$	29. $\left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	30. $m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{k}{n-m},$	
31. $\left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\langle\langle \begin{matrix} n \\ 0 \end{matrix} \rangle\rangle = 1,$	33. $\langle\langle \begin{matrix} n \\ k \end{matrix} \rangle\rangle = 0$ for $n \neq 0,$	
34. $\langle\langle \begin{matrix} n \\ k \end{matrix} \rangle\rangle = (k+1) \langle\langle \begin{matrix} n-1 \\ k \end{matrix} \rangle\rangle + (2n-1-k) \langle\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \rangle\rangle,$	35. $\sum_{k=0}^n \langle\langle \begin{matrix} n \\ k \end{matrix} \rangle\rangle = \frac{(2n)^n}{2^n},$		
36. $\left\{ \begin{matrix} x \\ x-n \end{matrix} \right\} = \sum_{k=0}^n \langle\langle \begin{matrix} n \\ k \end{matrix} \rangle\rangle \binom{x+n-1-k}{2n},$	37. $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \sum_{k=0}^n \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (m+1)^{n-k},$		

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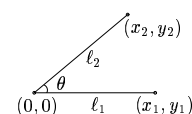
Identities Cont.		Trees
<p>38. $\binom{n+1}{m+1} = \sum_k \binom{n}{k} \binom{k}{m} = \sum_{k=0}^n \binom{k}{m} n^{\underline{n-k}} = n! \sum_{k=0}^n \frac{1}{k!} \binom{k}{m}$,</p> <p>40. $\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k}$,</p> <p>42. $\left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} = \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}$,</p> <p>44. $\binom{n}{m} = \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \binom{k}{m} (-1)^{m-k}$, 45. $(n-m)! \binom{n}{m} = \sum_k \binom{n+1}{k+1} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}$, for $n \geq m$,</p> <p>46. $\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \binom{m+k}{k}$,</p> <p>48. $\left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k}$,</p>	<p>39. $\left[\begin{matrix} x \\ x-n \end{matrix} \right] = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{2n}$,</p> <p>41. $\left[\begin{matrix} n \\ m \end{matrix} \right] = \sum_k \binom{n+1}{k+1} \binom{k}{m} (-1)^{m-k}$,</p> <p>43. $\left[\begin{matrix} m+n+1 \\ m \end{matrix} \right] = \sum_{k=0}^m k(n+k) \binom{n+k}{k}$,</p> <p>47. $\left[\begin{matrix} n \\ n-m \end{matrix} \right] = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\}$,</p> <p>49. $\left[\begin{matrix} n \\ \ell+m \end{matrix} \right] \binom{\ell+m}{\ell} = \sum_k \binom{k}{\ell} \left[\begin{matrix} n-k \\ m \end{matrix} \right] \binom{n}{k}$.</p>	<p>Every tree with n vertices has $n-1$ edges.</p> <p>Kraft inequality: If the depths of the leaves of a binary tree are d_1, \dots, d_n:</p> $\sum_{i=1}^n 2^{-d_i} \leq 1,$ <p>and equality holds only if every internal node has 2 sons.</p>
Recurrences		
<p>Master method: $T(n) = aT(n/b) + f(n)$, $a \geq 1, b > 1$ If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.</p> <p>If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \log_2 n)$.</p> <p>If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then $T(n) = \Theta(f(n))$.</p> <p>Substitution (example): Consider the following recurrence $T_{i+1} = 2^{2^i} \cdot T_i^2$, $T_1 = 2$.</p> <p>Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have $t_{i+1} = 2^i + 2t_i$, $t_1 = 1$.</p> <p>Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get $\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}$.</p> <p>Substituting we find $u_{i+1} = \frac{1}{2} + u_i$, $u_1 = \frac{1}{2}$,</p> <p>which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{2^{i-1}}$.</p> <p>Summing factors (example): Consider the following recurrence $T(n) = 3T(n/2) + n$, $T(1) = 1$.</p> <p>Rewrite so that all terms involving T are on the left side $T(n) - 3T(n/2) = n$.</p> <p>Now expand the recurrence, and choose a factor which makes the left side "telescope"</p>	$1(T(n) - 3T(n/2)) = n$ $3(T(n/2) - 3T(n/4)) = n/2$ \vdots $3^{\log_2 n - 1} (T(2) - 3T(1)) = 2$ <p>Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get $\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i$.</p> <p>Let $c = \frac{3}{2}$. Then we have $n \sum_{i=0}^{m-1} c^i = n \left(\frac{c^m - 1}{c - 1} \right)$ $= 2n(c^{\log_2 n} - 1)$ $= 2n(c^{(\log_2 n) \log_2 c} - 1)$ $= 2n^k - 2n$,</p> <p>and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider $T_i = 1 + \sum_{j=0}^{i-1} T_j$, $T_0 = 1$.</p> <p>Note that $T_{i+1} = 1 + \sum_{j=0}^i T_j$.</p> <p>Subtracting we find $T_{i+1} - T_i = 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j$ $= T_i$.</p> <p>And so $T_{i+1} = 2T_i = 2^{i+1}$.</p>	<p>Generating functions:</p> <ol style="list-style-type: none"> Multiply both sides of the equation by x^i. Sum both sides over all i for which the equation is valid. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$. Rewrite the equation in terms of the generating function $G(x)$. Solve for $G(x)$. The coefficient of x^i in $G(x)$ is g_i. <p>Example: $g_{i+1} = 2g_i + 1$, $g_0 = 0$.</p> <p>Multiply and sum: $\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i$.</p> <p>We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$: $\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i$.</p> <p>Simplify: $\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}$.</p> <p>Solve for $G(x)$: $G(x) = \frac{x}{(1-x)(1-2x)}$.</p> <p>Expand this using partial fractions: $G(x) = x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right)$ $= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right)$ $= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}$.</p> <p>So $g_i = 2^i - 1$.</p>

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$\pi \approx 3.14159,$		$e \approx 2.71828,$		$\gamma \approx 0.57721,$
		$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$		$\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.61803$
i	2^i	p_i	General	Probability
1	2	2	Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):	Continuous distributions: If
2	4	3	$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$	$\Pr[a < X < b] = \int_a^b p(x) dx,$
3	8	5	$B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	then p is the probability density function of X . If
4	16	7	Change of base, quadratic formula:	$\Pr[X < a] = P(a),$
5	32	11	$\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	then P is the distribution function of X . If P and p both exist then
6	64	13	Euler's number e :	$P(a) = \int_{-\infty}^a p(x) dx.$
7	128	17	$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$	Expectation: If X is discrete
8	256	19	$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	$E[g(X)] = \sum_x g(x) \Pr[X = x].$
9	512	23	$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$	If X continuous then
10	1,024	29	$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$	$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$
11	2,048	31	Harmonic numbers:	Variance, standard deviation:
12	4,096	37	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	$\text{VAR}[X] = E[X^2] - E[X]^2,$
13	8,192	41	$\ln n < H_n < \ln n + 1,$	$\sigma = \sqrt{\text{VAR}[X]}.$
14	16,384	43	$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$	For events A and B :
15	32,768	47	Factorial, Stirling's approximation:	$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$
16	65,536	53	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$
17	131,072	59	$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$	iff A and B are independent.
18	262,144	61	Ackermann's function and inverse:	$\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$
19	524,288	67	$a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$	For random variables X and Y :
20	1,048,576	71	$\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	$E[X \cdot Y] = E[X] \cdot E[Y],$
21	2,097,152	73	Binomial distribution:	if X and Y are independent.
22	4,194,304	79	$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$	$E[X + Y] = E[X] + E[Y],$
23	8,388,608	83	$E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$	$E[cX] = c E[X].$
24	16,777,216	89	Poisson distribution:	Bayes' theorem:
25	33,554,432	97	$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	$\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[B A_j] \Pr[A_j]}.$
26	67,108,864	101	Normal (Gaussian) distribution:	Inclusion-exclusion:
27	134,217,728	103	$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$
28	268,435,456	107	The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is	$\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$
29	536,870,912	109	$nH_n.$	Moment inequalities:
30	1,073,741,824	113		$\Pr[X \geq \lambda E[X]] \leq \frac{1}{\lambda},$
31	2,147,483,648	127		$\Pr[X - E[X] \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$
32	4,294,967,296	131		Geometric distribution:
Pascal's Triangle				$\Pr[X = k] = q^{k-1}p, \quad q = 1 - p,$
1				$E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$
1 1				
1 2 1				
1 3 3 1				
1 4 6 4 1				
1 5 10 10 5 1				
1 6 15 20 15 6 1				
1 7 21 35 35 21 7 1				
1 8 28 56 70 56 28 8 1				
1 9 36 84 126 126 84 36 9 1				
1 10 45 120 210 252 210 120 45 10 1				

Theoretical Computer Science Cheat Sheet

Trigonometry	Matrices	More Trig.																								
 <p>Pythagorean theorem: $C^2 = A^2 + B^2$.</p> <p>Definitions: $\sin a = A/C, \quad \cos a = B/C,$ $\csc a = C/A, \quad \sec a = C/B,$ $\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$</p> <p>Area, radius of inscribed circle: $\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$</p> <p>Identities: $\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$ $\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$ $1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$ $\sin x = \cos(\frac{\pi}{2} - x), \quad \sin x = \sin(\pi - x),$ $\cos x = -\cos(\pi - x), \quad \tan x = \cot(\frac{\pi}{2} - x),$ $\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{x}{2} - \cot x,$ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$ $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$ $\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$ $\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$ $\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$ $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$ $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y,$ $\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y.$</p> <p>Euler's equation: $e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$</p>	<p>Multiplication: $C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$</p> <p>Determinants: $\det A \neq 0$ iff A is non-singular. $\det A \cdot B = \det A \cdot \det B,$ $\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$</p> <p>$2 \times 2$ and 3×3 determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ $= ae i + b f g + c d h - c e g - f h a - i b d.$</p> <p>Permanents: $\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$</p>	 <p>Law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C.$</p> <p>Area: $A = \frac{1}{2}bc,$ $= \frac{1}{2}ab \sin C,$ $= \frac{c^2 \sin A \sin B}{2 \sin C}.$</p> <p>Heron's formula: $A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$ $s = \frac{1}{2}(a + b + c),$ $s_a = s - a,$ $s_b = s - b,$ $s_c = s - c.$</p>																								
<p>©1994 by Steve Seiden sseiden@acm.org http://www.mpi-sb.mpg.de/~seiden</p>	<p>Hyperbolic Functions</p> <p>Definitions: $\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$ $\text{sech } x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$</p> <p>Identities: $\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$ $\coth^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$ $\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$ $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$ $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$ $\sinh 2x = 2 \sinh x \cosh x,$ $\cosh 2x = \cosh^2 x + \sinh^2 x,$ $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$ $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$ $2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$</p> <table border="1" data-bbox="877 1534 1093 1713"> <thead> <tr> <th>θ</th> <th>$\sin \theta$</th> <th>$\cos \theta$</th> <th>$\tan \theta$</th> </tr> </thead> <tbody> <tr> <td>0</td> <td>0</td> <td>1</td> <td>0</td> </tr> <tr> <td>$\frac{\pi}{6}$</td> <td>$\frac{1}{2}$</td> <td>$\frac{\sqrt{3}}{2}$</td> <td>$\frac{\sqrt{3}}{3}$</td> </tr> <tr> <td>$\frac{\pi}{4}$</td> <td>$\frac{\sqrt{2}}{2}$</td> <td>$\frac{\sqrt{2}}{2}$</td> <td>1</td> </tr> <tr> <td>$\frac{\pi}{3}$</td> <td>$\frac{\sqrt{3}}{2}$</td> <td>$\frac{1}{2}$</td> <td>$\sqrt{3}$</td> </tr> <tr> <td>$\frac{\pi}{2}$</td> <td>1</td> <td>0</td> <td>∞</td> </tr> </tbody> </table> <p>... in mathematics you don't understand things, you just get used to them. - J. von Neumann</p>	θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	0	0	1	0	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\pi}{2}$	1	0	∞	<p>More identities: $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$ $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$ $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$ $= \frac{1 - \cos x}{\sin x},$ $= \frac{\sin x}{1 + \cos x},$ $\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$ $= \frac{1 + \cos x}{\sin x},$ $= \frac{\sin x}{1 - \cos x},$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i},$ $\cos x = \frac{e^{ix} + e^{-ix}}{2},$ $\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$ $= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$ $\sin x = \frac{\sinh ix}{i},$ $\cos x = \cosh ix,$ $\tan x = \frac{\tanh ix}{i}.$</p>
θ	$\sin \theta$	$\cos \theta$	$\tan \theta$																							
0	0	1	0																							
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$																							
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1																							
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$																							
$\frac{\pi}{2}$	1	0	∞																							

Theoretical Computer Science Cheat Sheet

Number Theory	Graph Theory									
<p>The Chinese remainder theorem: There exists a number C such that:</p> $C \equiv r_1 \pmod{m_1}$ \vdots $C \equiv r_n \pmod{m_n}$ <p>if m_i and m_j are relatively prime for $i \neq j$.</p> <p>Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then</p> $\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$ <p>Euler's theorem: If a and b are relatively prime then</p> $1 \equiv a^{\phi(b)} \pmod{b}.$ <p>Fermat's theorem:</p> $1 \equiv a^{p-1} \pmod{p}.$ <p>The Euclidean algorithm: if $a > b$ are integers then</p> $\gcd(a, b) = \gcd(a \bmod b, b).$ <p>If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then</p> $S(x) = \sum_{d x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$ <p>Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.</p> <p>Wilson's theorem: n is a prime iff</p> $(n-1)! \equiv -1 \pmod{n}.$ <p>Möbius inversion:</p> $\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$ <p>If</p> $G(a) = \sum_{d a} F(d),$ <p>then</p> $F(a) = \sum_{d a} \mu(d) G\left(\frac{a}{d}\right).$ <p>Prime numbers:</p> $p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$ $\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$	<p>Definitions:</p> <p><i>Loop</i> An edge connecting a vertex to itself.</p> <p><i>Directed Simple</i> Each edge has a direction. Graph with no loops or multi-edges.</p> <p><i>Walk</i> A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.</p> <p><i>Trail</i> A walk with distinct edges.</p> <p><i>Path</i> A trail with distinct vertices.</p> <p><i>Connected</i> A graph where there exists a path between any two vertices.</p> <p><i>Component</i> A maximal connected subgraph.</p> <p><i>Tree</i> A connected acyclic graph.</p> <p><i>Free tree</i> A tree with no root.</p> <p><i>DAG</i> Directed acyclic graph.</p> <p><i>Eulerian</i> Graph with a trail visiting each edge exactly once.</p> <p><i>Hamiltonian</i> Graph with a cycle visiting each vertex exactly once.</p> <p><i>Cut</i> A set of edges whose removal increases the number of components.</p> <p><i>Cut-set</i> A minimal cut.</p> <p><i>Cut edge</i> A size 1 cut.</p> <p><i>k-Connected</i> A graph connected with the removal of any $k-1$ vertices.</p> <p><i>k-Tough</i> $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G-S) \leq S$.</p> <p><i>k-Regular</i> A graph where all vertices have degree k.</p> <p><i>k-Factor</i> A k-regular spanning subgraph.</p> <p><i>Matching</i> A set of edges, no two of which are adjacent.</p> <p><i>Clique</i> A set of vertices, all of which are adjacent.</p> <p><i>Ind. set</i> A set of vertices, none of which are adjacent.</p> <p><i>Vertex cover</i> A set of vertices which cover all edges.</p> <p><i>Planar graph</i> A graph which can be embedded in the plane.</p> <p><i>Plane graph</i> An embedding of a planar graph.</p> <hr/> $\sum_{v \in V} \deg(v) = 2m.$ <p>If G is planar then $n - m + f = 2$, so $f \leq 2n - 4$, $m \leq 3n - 6$.</p> <p>Any planar graph has a vertex with degree ≤ 5.</p>	<p>Notation:</p> <p>$E(G)$ Edge set</p> <p>$V(G)$ Vertex set</p> <p>$c(G)$ Number of components</p> <p>$G[S]$ Induced subgraph</p> <p>$\deg(v)$ Degree of v</p> <p>$\Delta(G)$ Maximum degree</p> <p>$\delta(G)$ Minimum degree</p> <p>$\chi(G)$ Chromatic number</p> <p>$\chi_E(G)$ Edge chromatic number</p> <p>G^c Complement graph</p> <p>K_n Complete graph</p> <p>K_{n_1, n_2} Complete bipartite graph</p> <p>$r(k, \ell)$ Ramsey number</p> <hr/> <p style="text-align: center;">Geometry</p> <p>Projective coordinates: triples (x, y, z), not all x, y and z zero.</p> $(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$ <table style="width: 100%; border-collapse: collapse;"> <thead> <tr> <th style="width: 50%; text-align: center;">Cartesian</th> <th style="width: 50%; text-align: center;">Projective</th> </tr> </thead> <tbody> <tr> <td>(x, y)</td> <td>$(x, y, 1)$</td> </tr> <tr> <td>$y = mx + b$</td> <td>$(m, -1, b)$</td> </tr> <tr> <td>$x = c$</td> <td>$(1, 0, -c)$</td> </tr> </tbody> </table> <p>Distance formula, L_p and L_∞ metric:</p> $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$ $[x_1 - x_0 ^p + y_1 - y_0 ^p]^{1/p},$ $\lim_{p \rightarrow \infty} [x_1 - x_0 ^p + y_1 - y_0 ^p]^{1/p}.$ <p>Area of triangle $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2):</p> $\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$ <p>Angle formed by three points:</p>  $\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{l_1 l_2}.$ <p>Line through two points (x_0, y_0) and (x_1, y_1):</p> $\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$ <p>Area of circle, volume of sphere:</p> $A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$	Cartesian	Projective	(x, y)	$(x, y, 1)$	$y = mx + b$	$(m, -1, b)$	$x = c$	$(1, 0, -c)$
Cartesian	Projective									
(x, y)	$(x, y, 1)$									
$y = mx + b$	$(m, -1, b)$									
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		<p>If I have seen farther than others, it is because I have stood on the shoulders of giants.</p> <p style="text-align: right;">- Issac Newton</p>								

Theoretical Computer Science Cheat Sheet

π	Calculus
<p>Wallis' identity: $\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$</p> <p>Brouncker's continued fraction expansion: $\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{\ddots}}}}$</p> <p>Gregory's series: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$</p> <p>Newton's series: $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$</p> <p>Sharp's series: $\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$</p> <p>Euler's series: $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$ $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$ $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$</p>	<p>Derivatives:</p> <ol style="list-style-type: none"> $\frac{d(cu)}{dx} = c \frac{du}{dx}$ $\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ $\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$ $\frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}$ $\frac{d(u/v)}{dx} = \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2}$ $\frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx}$ $\frac{d(c^u)}{dx} = (\ln c)c^u \frac{du}{dx}$ $\frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx}$ $\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$ $\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$ $\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$ $\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$ $\frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx}$ $\frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx}$ $\frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$ $\frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$ $\frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$ $\frac{d(\text{arccot } u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx}$ $\frac{d(\text{arcsec } u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}$ $\frac{d(\text{arccsc } u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}$ $\frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx}$ $\frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx}$ $\frac{d(\tanh u)}{dx} = \text{sech}^2 u \frac{du}{dx}$ $\frac{d(\coth u)}{dx} = -\text{csch}^2 u \frac{du}{dx}$ $\frac{d(\text{sech } u)}{dx} = -\text{sech } u \tanh u \frac{du}{dx}$ $\frac{d(\text{csch } u)}{dx} = -\text{csch } u \coth u \frac{du}{dx}$ $\frac{d(\arcsinh u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$ $\frac{d(\text{arccosh } u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$ $\frac{d(\text{arctanh } u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$ $\frac{d(\text{arcoth } u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx}$ $\frac{d(\text{arcsech } u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}$ $\frac{d(\text{arccsch } u)}{dx} = \frac{-1}{ u \sqrt{1+u^2}} \frac{du}{dx}$ <p>Integrals:</p> <ol style="list-style-type: none"> $\int cu \, dx = c \int u \, dx$ $\int (u+v) \, dx = \int u \, dx + \int v \, dx$ $\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1$ $\int \frac{1}{x} \, dx = \ln x$ $\int e^x \, dx = e^x$ $\int \frac{dx}{1+x^2} = \arctan x$ $\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$ $\int \sin x \, dx = -\cos x$ $\int \cos x \, dx = \sin x$ $\int \tan x \, dx = -\ln \cos x$ $\int \cot x \, dx = \ln \cos x$ $\int \sec x \, dx = \ln \sec x + \tan x$ $\int \csc x \, dx = \ln \csc x + \cot x$ $\int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$
<p style="text-align: center;">Partial Fractions</p> <p>Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining</p> $\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)}$ <p>where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:</p> $\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)}$ <p>where</p> $A = \left[\frac{N(x)}{D(x)} \right]_{x=a}$ <p>For a repeated factor:</p> $\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)}$ <p>where</p> $A_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \left(\frac{N(x)}{D(x)} \right) \right]_{x=a}$	
<p>The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable. - George Bernard Shaw</p>	

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}$, $a > 0$, 16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2)$, $a > 0$,
17. $\int \sin^2(ax) dx = \frac{1}{2a}(ax - \sin(ax) \cos(ax))$, 18. $\int \cos^2(ax) dx = \frac{1}{2a}(ax + \sin(ax) \cos(ax))$,
19. $\int \sec^2 x dx = \tan x$, 20. $\int \csc^2 x dx = -\cot x$,
21. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$, 22. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$,
23. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$, $n \neq 1$, 24. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$, $n \neq 1$,
25. $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx$, $n \neq 1$,
26. $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx$, $n \neq 1$, 27. $\int \sinh x dx = \cosh x$, 28. $\int \cosh x dx = \sinh x$,
29. $\int \tanh x dx = \ln |\cosh x|$, 30. $\int \coth x dx = \ln |\sinh x|$, 31. $\int \operatorname{sech} x dx = \arctan \sinh x$, 32. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|$,
33. $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x$, 34. $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x$, 35. $\int \operatorname{sech}^2 x dx = \tanh x$,
36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}$, $a > 0$, 37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|$,
38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2})$, $a > 0$,
40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}$, $a > 0$, 41. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$, $a > 0$,
42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}$, $a > 0$,
43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}$, $a > 0$, 44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|$, 45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}$,
46. $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|$, 47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|$, $a > 0$,
48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|$, 49. $\int x\sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2}$,
50. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx$, 51. $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|$, $a > 0$,
52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|$, 53. $\int x\sqrt{a^2 - x^2} dx = -\frac{1}{3}(a^2 - x^2)^{3/2}$,
54. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}$, $a > 0$, 55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|$,
56. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$, 57. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}$, $a > 0$,
58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|$, 59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}$, $a > 0$,
60. $\int x\sqrt{x^2 \pm a^2} dx = \frac{1}{3}(x^2 \pm a^2)^{3/2}$, 61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|$,

Theoretical Computer Science Cheat Sheet

Calculus Cont.

62. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \arccos \frac{a}{|x|}, \quad a > 0,$ 63. $\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x},$
 64. $\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2},$ 65. $\int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx = \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3},$
 66. $\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases}$
 67. $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases}$
 68. $\int \sqrt{ax^2 + bx + c} dx = \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$
 69. $\int \frac{x dx}{\sqrt{ax^2 + bx + c}} = \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}},$
 70. $\int \frac{dx}{x\sqrt{ax^2 + bx + c}} = \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases}$
 71. $\int x^3 \sqrt{x^2 + a^2} dx = \left(\frac{1}{5}x^2 - \frac{2}{15}a^2\right)(x^2 + a^2)^{3/2},$
 72. $\int x^n \sin(ax) dx = -\frac{1}{a}x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx,$
 73. $\int x^n \cos(ax) dx = \frac{1}{a}x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx,$
 74. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx,$
 75. $\int x^n \ln(ax) dx = x^{n+1} \left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right),$
 76. $\int x^n (\ln ax)^m dx = \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.$

Finite Calculus

Difference, shift operators:
 $\Delta f(x) = f(x+1) - f(x),$
 $E f(x) = f(x+1).$
 Fundamental Theorem:
 $f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x) + C.$
 $\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$
 Differences:
 $\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$
 $\Delta(uv) = u\Delta v + E v \Delta u,$
 $\Delta(x^n) = nx^{n-1},$
 $\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$
 $\Delta(c^x) = (c-1)c^x, \quad \Delta\binom{x}{m} = \binom{x}{m-1}.$
 Sums:
 $\sum cu \delta x = c \sum u \delta x,$
 $\sum(u+v) \delta x = \sum u \delta x + \sum v \delta x,$
 $\sum u \Delta v \delta x = uv - \sum E v \Delta u \delta x,$
 $\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$
 $\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m-1}.$
 Falling Factorial Powers:
 $x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n > 0,$
 $x^{\underline{0}} = 1,$
 $x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n < 0,$
 $x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$
 Rising Factorial Powers:
 $x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n > 0,$
 $x^{\overline{0}} = 1,$
 $x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n < 0,$
 $x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{1}} = x^1 = x^{\overline{1}}$
 $x^{\underline{2}} = x^2 + x^1 = x^{\overline{2}} - x^{\overline{1}}$
 $x^{\underline{3}} = x^3 + 3x^2 + x^1 = x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$
 $x^{\underline{4}} = x^4 + 6x^3 + 7x^2 + x^1 = x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$
 $x^{\underline{5}} = x^5 + 15x^4 + 25x^3 + 10x^2 + x^1 = x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$
 $x^{\overline{1}} = x^1 = x^{\underline{1}}$
 $x^{\overline{2}} = x^2 + x^1 = x^{\underline{2}} - x^{\underline{1}}$
 $x^{\overline{3}} = x^3 + 3x^2 + 2x^1 = x^{\underline{3}} - 3x^{\underline{2}} + 2x^{\underline{1}}$
 $x^{\overline{4}} = x^4 + 6x^3 + 11x^2 + 6x^1 = x^{\underline{4}} - 6x^{\underline{3}} + 11x^{\underline{2}} - 6x^{\underline{1}}$
 $x^{\overline{5}} = x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 = x^{\underline{5}} - 10x^{\underline{4}} + 35x^{\underline{3}} - 50x^{\underline{2}} + 24x^{\underline{1}}$

$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
 $= 1/(x+1)^{\overline{-n}},$
 $x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
 $= 1/(x-1)^{\underline{-n}},$
 $x^n = \sum_{k=1}^n \binom{n}{k} x^k = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^{\overline{k}},$
 $x^{\underline{n}} = \sum_{k=1}^n \binom{n}{k} (-1)^{n-k} x^k,$
 $x^{\overline{n}} = \sum_{k=1}^n \binom{n}{k} x^k.$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$\frac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \dots$	$= \sum_{i=0}^{\infty} x^i,$
$\frac{1}{1-cx}$	$= 1 + cx + c^2x^2 + c^3x^3 + \dots$	$= \sum_{i=0}^{\infty} c^i x^i,$
$\frac{1}{1-x^n}$	$= 1 + x^n + x^{2n} + x^{3n} + \dots$	$= \sum_{i=0}^{\infty} x^{ni},$
$\frac{x}{(1-x)^2}$	$= x + 2x^2 + 3x^3 + 4x^4 + \dots$	$= \sum_{i=0}^{\infty} ix^i,$
$x^k \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right)$	$= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots$	$= \sum_{i=0}^{\infty} i^n x^i,$
e^x	$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{x^i}{i!},$
$\ln(1+x)$	$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$	$= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$
$\ln \frac{1}{1-x}$	$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$	$= \sum_{i=1}^{\infty} \frac{x^i}{i},$
$\sin x$	$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$
$\cos x$	$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$
$\tan^{-1} x$	$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$	$= \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$
$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$
$\frac{1}{(1-x)^{n+1}}$	$= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$
$\frac{x}{e^x - 1}$	$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots$	$= \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$
$\frac{1}{2x}(1 - \sqrt{1-4x})$	$= 1 + x + 2x^2 + 5x^3 + \dots$	$= \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}}$	$= 1 + x + 2x^2 + 6x^3 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$
$\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n$	$= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots$	$= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$
$\frac{1}{1-x} \ln \frac{1}{1-x}$	$= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{23}{12}x^4 + \dots$	$= \sum_{i=1}^{\infty} H_i x^i,$
$\frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2$	$= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots$	$= \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$
$\frac{x}{1-x-x^2}$	$= x + x^2 + 2x^3 + 3x^4 + \dots$	$= \sum_{i=0}^{\infty} F_i x^i,$
$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$	$= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots$	$= \sum_{i=0}^{\infty} F_{ni} x^i.$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$x A'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^i a_j$ then

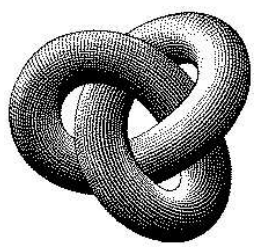
$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
- Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series		Escher's Knot
<p>Expansions:</p> $\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} = \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,$ $x^{\bar{n}} = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$ $\left(\ln \frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty} \binom{n}{i} \frac{n! x^i}{i!},$ $\tan x = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!},$ $\zeta(x) = \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$ $\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$ $\zeta^2(x) = \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \text{ where } d(n) = \sum_{d n} 1,$ $\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \text{ where } S(n) = \sum_{d n} d,$ $\zeta(2n) = \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!}, \quad n \in \mathbb{N},$ $\frac{x}{\sin x} = \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4i-2) B_{2i} x^{2i}}{(2i)!},$ $\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$ $e^x \sin x = \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,$ $\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)!(2i+1)!} x^i,$ $\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$		
<p>Stieltjes Integration</p> <p>If G is continuous in the interval $[a, b]$ and F is nondecreasing then</p> $\int_a^b G(x) dF(x)$ <p>exists. If $a \leq b \leq c$ then</p> $\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$ <p>If the integrals involved exist</p> $\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$ $\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$ $\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$ $\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$ <p>If the integrals involved exist, and F possesses a derivative F' at every point in $[a, b]$ then</p> $\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$		
<p>Cramer's Rule</p> <p>If we have equations:</p> $a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1$ $a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n = b_2$ \vdots $a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n = b_n$ <p>Let $A = (a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B. Then</p> $x_i = \frac{\det A_i}{\det A}.$		<p>Fibonacci Numbers</p> <p>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...</p> <p>Definitions:</p> $F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$ $F_{-i} = (-1)^{i-1} F_i,$ $F_i = \frac{1}{\sqrt{5}} (\phi^i - \hat{\phi}^i),$ <p>Cassini's identity: for $i > 0$:</p> $F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$ <p>Additive rule:</p> $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$ $F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$ <p>Calculation by matrices:</p> $\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$
<p>Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius. - William Blake (The Marriage of Heaven and Hell)</p>		<p>The Fibonacci number system:</p> <p>Every integer n has a unique representation</p> $n = F_{k_1} + F_{k_2} + \dots + F_{k_m},$ <p>where $k_i \geq k_{i+1} + 2$ for all i, $1 \leq i < m$ and $k_m \geq 2$.</p>