

13.2 Relabel to front

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 48 discharge(u)

```
1: while  $u$  is active do
2:    $v \leftarrow u.current\text{-}neighbour$ 
3:   if  $v = \text{null}$  then
4:     relabel( $u$ )
5:      $u.current\text{-}neighbour \leftarrow u.neighbour\text{-}list\text{-}head$ 
6:   else
7:     if  $(u, v)$  admissable then push( $u, v$ )
8:     else  $u.current\text{-}neighbour \leftarrow v.next\text{-}in\text{-}list$ 
```

13.2 Relabel to front

Lemma 73

If $v = \text{null}$ in line 3, then there is no outgoing admissible edge from u .

The lemma holds because push- and relabel-operations on nodes different from u cannot make edges outgoing from u admissible.

This shows that $\text{discharge}(u)$ is correct, and that we can perform a relabel in line 4.

13.2 Relabel to front

Algorithm 49 relabel-to-front(G, s, t)

```
1: initialize preflow
2: initialize node list  $L$  containing  $V \setminus \{s, t\}$  in any order
3: foreach  $u \in V \setminus \{s, t\}$  do
4:    $u.current\text{-neighbour} \leftarrow u.neighbour\text{-list}\text{-head}$ 
5:  $u \leftarrow L.head$ 
6: while  $u \neq \text{null}$  do
7:    $old\text{-height} \leftarrow \ell(u)$ 
8:   discharge( $u$ )
9:   if  $\ell(u) > old\text{-height}$  then
10:     move  $u$  to the front of  $L$ 
11:    $u \leftarrow u.next$ 
```

13.2 Relabel to front

Lemma 74 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

- 1. The sequence L is topologically sorted w.r.t. the set of admissible edges; this means for an admissible edge (x, y) the node x appears before y in sequence L .*
- 2. No node before u in the list L is active.*

Proof:

► Initialization:

1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissible, which means that any ordering L is permitted.
2. We start with u being the head of the list; hence no node before u can be active

► Maintenance:

1.
 - Pushes do not create any new admissible edges. Therefore, not relabeling u leaves L topologically sorted.
 - After relabeling, u cannot have admissible incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissible edges leaving u that were generated by the relabeling.

13.2 Relabel to front

Proof:

► Maintenance:

2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u ; the $\text{discharge}(u)$ operation only terminates when u is not active anymore.

For the case that we do a relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissible arc. However, all admissible arcs point to successors of u .

Note that the invariant for $u = \text{null}$ means that we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

13.2 Relabel to front

Lemma 75

There are at most $\mathcal{O}(n^3)$ calls to $\text{discharge}(u)$.

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#\text{relabels} + 1) = \mathcal{O}(n^3)$.

13.2 Relabel to front

Lemma 76

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting *u.current-neighbour*). In total we have $\mathcal{O}(n^2)$ relabel-operations.

13.2 Relabel to front

Note that by definition a saturating push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 77

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.current-neighbour$.

This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $degree(u) + 1$ many entries (+1 for null-entry).

13.2 Relabel to front

Lemma 78

The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to `discharge()`. Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 79

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.