17 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m\,\mathrm{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

18 Augmenting Paths for Matchings

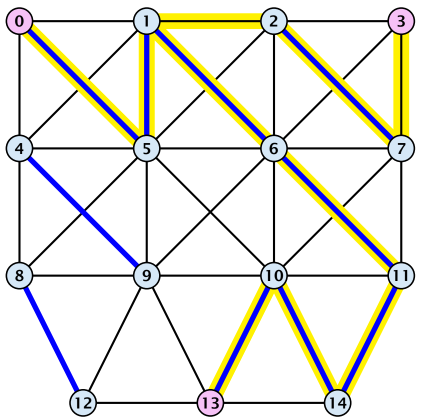
Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
- An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

Theorem 1

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

Augmenting Paths in Action



18 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

18 Augmenting Paths for Matchings

Algorithmic idea:

As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 2

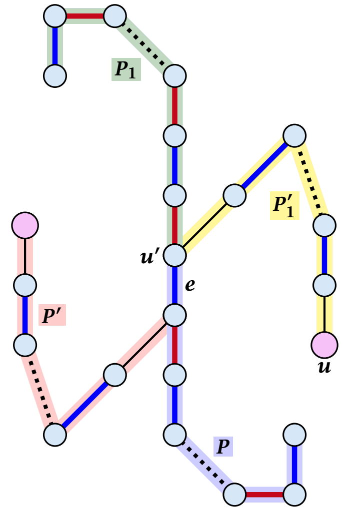
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

The above theorem allows for an easier implementation of an augmenting path algorithm. Once we checked for augmenting paths starting from u we don't have to check for such paths in future rounds.

18 Augmenting Paths for Matchings

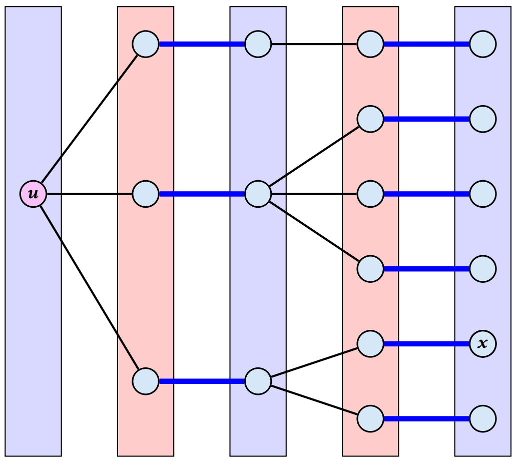
Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P_1'.
- $P_1 \circ P_1'$ is augmenting path in M (↯).



How to find an augmenting path?

Construct an alternating tree.



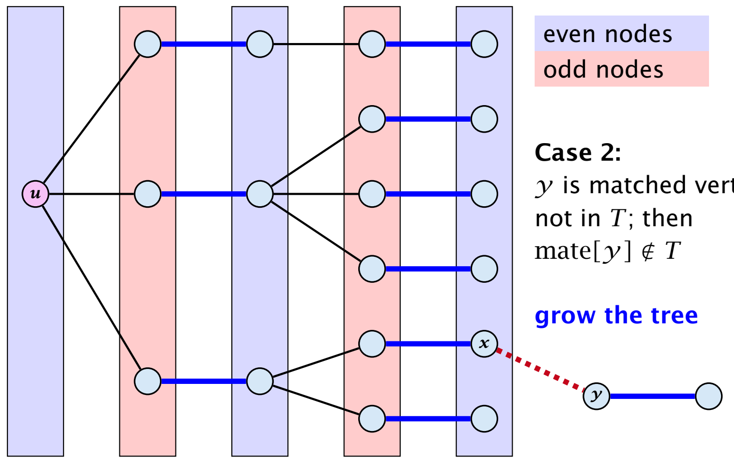
even nodes
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.



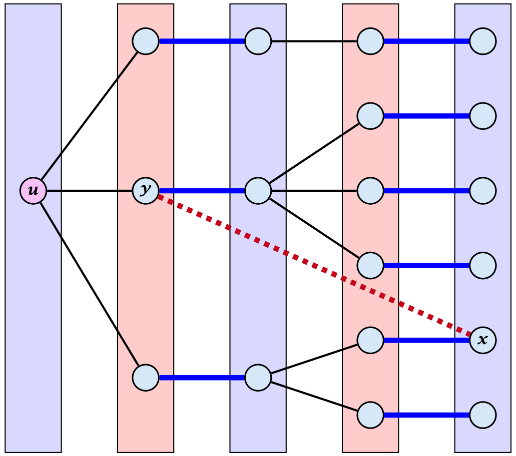
even nodes
odd nodes

Case 2:
y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



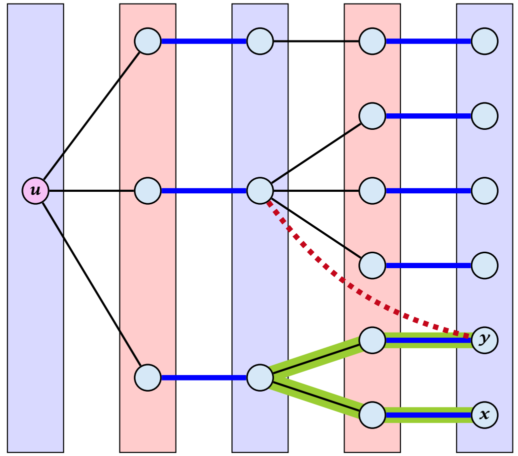
even nodes
odd nodes

Case 3:
y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 50 BiMatch$(G, match)$

for $x \in V$ **do** $mate[x] \leftarrow 0$;
$r \leftarrow 0$; $free \leftarrow n$;
while $free \geq 1$ **and** $r < n$ **do**
  $r \leftarrow r + 1$
  if $mate[r] = 0$ **then**
    for $i = 1$ **to** m **do** $parent[i'] \leftarrow 0$
    $Q \leftarrow \emptyset$; $Q.\text{append}(r)$; $aug \leftarrow \text{false}$;
    while $aug = \text{false}$ **and** $Q \neq \emptyset$ **do**
      $x \leftarrow Q.\text{dequeue}()$;
      for $y \in A_x$ **do**
        if $mate[y] = 0$ **then**
          $\text{augm}(mate, parent, y)$;
          $aug \leftarrow \text{true}$;
          $free \leftarrow free - 1$;
        else
          if $parent[y] = 0$ **then**
            $parent[y] \leftarrow x$;
            $Q.\text{enqueue}(mate[y])$;

graph $G = (S \cup S', E)$

$S = \{1, \ldots, n\}$

$S' = \{1', \ldots, n'\}$

start with an empty matching

free: number of unmatched nodes in S

r: root of current tree

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

r is the new node that

19 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$