## Integer Multicommodity Flows

Choose $\delta = \sqrt{(c \ln n)/W^*}$.

Then

$$\Pr[Y_e \geq (1+\delta)W^*] < e^{-W^*\delta^2/3} = \frac{1}{n^{c/3}}$$



## Repetition: Primal Dual for Set Cover

**Primal Relaxation:**

$$\begin{array}{lrrcl} \min & & \sum_{i=1}^{k} w_i x_i & & \\ \text{s.t.} & \forall u \in U & \sum_{i:u \in S_i} x_i & \geq & 1 \\ & \forall i \in \{1,\dots,k\} & x_i & \geq & 0 \end{array}$$

**Dual Formulation:**

$$\begin{array}{lrrcl} \max & & \sum_{u \in U} y_u & & \\ \text{s.t.} & \forall i \in \{1,\dots,k\} & \sum_{u:u \in S_i} y_u & \leq & w_i \\ & & y_u & \geq & 0 \end{array}$$



## Repetition: Primal Dual for Set Cover

**Algorithm:**

- Start with $y = 0$ (feasible dual solution).
  Start with $x = 0$ (integral primal solution that may be infeasible).
- While $x$ not feasible
  - Identify an element $e$ that is not covered in current primal integral solution.
  - Increase dual variable $y_e$ until a dual constraint becomes tight (maybe increase by $0$!).
  - If this is the constraint for set $S_j$ set $x_j = 1$ (add this set to your solution).



## Repetition: Primal Dual for Set Cover

**Analysis:**

- For every set $S_j$ with $x_j = 1$ we have

$$\sum_{e \in S_j} y_e = w_j$$

- Hence our cost is

$$\sum_j w_j = \sum_j \sum_{e \in S_j} y_e = \sum_e |\{j : e \in S_j\}| \cdot y_e \leq f \cdot \sum_e y_e \leq f \cdot \mathrm{OPT}$$

Note that the constructed pair of primal and dual solution fulfills primal slackness conditions.

This means

$$x_j > 0 \Rightarrow \sum_{e \in S_j} y_e = w_j$$

If we would also fulfill dual slackness conditions

$$y_e > 0 \Rightarrow \sum_{j: e \in S_j} x_j = 1$$

then the solution would be optimal!!!



We don't fulfill these constraint but we fulfill an approximate version:

$$y_e > 0 \Rightarrow 1 \le \sum_{j: e \in S_j} x_j \le f$$

This is sufficient to show that the solution is an $f$-approximation.



Suppose we have a primal/dual pair

$$\begin{array}{llrcl} \min & & \sum_j c_j x_j & & \\ \text{s.t.} & \forall i & \sum_{j:} a_{ij} x_j & \ge & b_i \\ & \forall j & x_j & \ge & 0 \end{array} \qquad \begin{array}{llrcl} \max & & \sum_i b_i y_i & & \\ \text{s.t.} & \forall j & \sum_i a_{ij} y_i & \le & c_j \\ & \forall i & y_i & \ge & 0 \end{array}$$

and solutions that fulfill approximate slackness conditions:

$$x_j > 0 \Rightarrow \sum_i a_{ij} y_i \ge \frac{1}{\alpha} c_j$$

$$y_i > 0 \Rightarrow \sum_j a_{ij} x_j \le \beta b_i$$



Then

right hand side of $j$-th dual constraint

$$\underbrace{\boxed{\sum_j \boxed{c_j} x_j}}_{\text{primal cost}} \le \alpha \sum_j \left( \sum_i a_{ij} y_i \right) x_j$$

$$= \alpha \sum_i \left( \sum_j a_{ij} x_j \right) y_i$$

$$\le \alpha\beta \cdot \underbrace{\boxed{\sum_i b_i y_i}}_{\text{dual objective}}$$

## Feedback Vertex Set for Undirected Graphs

- Given a graph $G = (V, E)$ and non-negative weights $w_v \geq 0$ for vertex $v \in V$.
- Choose a minimum cost subset of vertices s.t. every cycle contains at least one vertex.



We can encode this as an instance of Set Cover

- Each vertex can be viewed as a set that contains some cycles.
- However, this encoding gives a Set Cover instance of non-polynomial size.
- The $O(\log n)$-approximation for Set Cover does not help us to get a good solution.



Let $C$ denote the set of all cycles (where a cycle is identified by its set of vertices)

**Primal Relaxation:**

$$\begin{array}{llrcl}
\min & & \sum_v w_v x_v & & \\
\text{s.t.} & \forall C \in C & \sum_{v \in C} x_v & \geq & 1 \\
& \forall v & x_v & \geq & 0
\end{array}$$

**Dual Formulation:**

$$\begin{array}{llrcl}
\max & & \sum_{C \in C} y_C & & \\
\text{s.t.} & \forall v \in V & \sum_{C : v \in C} y_C & \leq & w_v \\
& \forall C & y_C & \geq & 0
\end{array}$$



If we perform the previous dual technique for Set Cover we get the following:

- Start with $x = 0$ and $y = 0$
- While there is a cycle $C$ that is not covered (does not contain a chosen vertex).
  - Increase $y_e$ until dual constraint for some vertex $v$ becomes tight.
  - set $x_v = 1$.

Then

$$\begin{aligned}\sum_v w_v x_v &= \sum_v \sum_{C:v\in C} y_C x_v \\ &= \sum_{v\in S} \sum_{C:v\in C} y_C \\ &= \sum_C |S\cap C|\cdot y_C\end{aligned}$$

where $S$ is the set of vertices we choose.

If every cycle is short we get a good approximation ratio, but this is unrealistic.



**Algorithm 1** FeedbackVertexSet

$y \leftarrow 0$
$x \leftarrow 0$
**while** exists cycle $C$ in $G$ **do**
  increase $y_C$ until there is $v \in C$ s.t. $\sum_{C:v\in C} y_C = w_v$
  $x_v = 1$
  remove $v$ from $G$
  repeatedly remove vertices of degree 1 from $G$



**Idea:**
Always choose a short cycle that is not covered. If we always find a cycle of length at most $\alpha$ we get an $\alpha$-approximation.

**Observation:**
For any path $P$ of vertices of degree $2$ in $G$ the algorithm chooses at most one vertex from $P$.



**Observation:**
If we always choose a cycle for which the number of vertices of degree at least 3 is at most $\alpha$ we get an $\alpha$-approximation.

**Theorem 15**
*In any graph with no vertices of degree 1, there always exists a cycle that has at most $\mathcal{O}(\log n)$ vertices of degree 3 or more. We can find such a cycle in linear time.*

This means we have

$$y_C > 0 \Rightarrow |S\cap C| \le \mathcal{O}(\log n) \ .$$

## Primal Dual for Shortest Path

Given a graph $G = (V, E)$ with two nodes $s, t \in V$ and edge-weights $c : E \to \mathbb{R}^+$ find a shortest path between $s$ and $t$ w.r.t. edge-weights $c$.

$$\begin{array}{llrcl} \min & & \sum_e c(e) x_e & & \\ \text{s.t.} & \forall S \in \mathcal{S} & \sum_{e:\delta(S)} x_e & \ge & 1 \\ & \forall e \in E & x_e & \in & \{0,1\} \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in $S$, and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.



## Primal Dual for Shortest Path

**The Dual:**

$$\begin{array}{llrcl} \max & & \sum_S y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S: e \in \delta(S)} y_S & \le & c(e) \\ & \forall S \in \mathcal{S} & y_S & \ge & 0 \end{array}$$

Here $\delta(S)$ denotes the set of edges with exactly one end-point in $S$, and $\mathcal{S} = \{S \subseteq V : s \in S, t \notin S\}$.



## Primal Dual for Shortest Path

We can interpret the value $y_S$ as the width of a moat surounding the set $S$.

Each set can have its own moat but all moats must be disjoint.

An edge cannot be shorter than all the moats that it has to cross.



**Algorithm 1** PrimalDualShortestPath

$y \leftarrow 0$
$F \leftarrow \emptyset$
**while** there is no $s$-$t$ path in $(V, F)$ **do**
    Let $C$ be the connected component of $(V, F)$ containing $s$
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ such that $\sum_{S: e' \in \delta(S)} y_S = c(e')$.
    $F \leftarrow F \cup \{e'\}$
Let $P$ be an $s$-$t$ path in $(V, F)$
**return** $P$

**Lemma 16**
*At each point in time the set $F$ forms a tree.*

**Proof:**
- In each iteration we take the current connected component from $(V,F)$ that contains $s$ (call this component $C$) and add some edge from $\delta(C)$ to $F$.
- Since, at most one end-point of the new edge is in $C$ the edge cannot close a cycle.



$$
\begin{aligned}
\sum_{e\in P} c_(e) &= \sum_{e\in P}\sum_{S:e\in\delta(S)} y_S\\
&= \sum_{S:s\in S,t\notin S} |P\cap\delta(S)|\cdot y_S\ .
\end{aligned}
$$

If we can show that $y_S > 0$ implies $|P\cap\delta(S)| = 1$ gives

$$
\sum_{e\in P} c(e) = \sum_S y_S \le \mathrm{OPT}
$$

by weak duality.

Hence, we find a shortest path.



If $S$ contains two edges from $P$ then there must exist a subpath $P'$ of $P$ that starts and ends with a vertex from $S$ (and all interior vertices are not in $S$).

When we increased $y_S$, $S$ was a connected component of the set of edges $F'$ that we had chosen till this point.

$F'\cup P'$ contains a cycle. Hence, also the final set of edges contains a cycle.

This is a contradiction.



**Steiner Forest Problem:**
Given a graph $G=(V,E)$, together with source-target pairs $s_i,t_i,i=1,\dots,k$, and a cost function $c:E\to\mathbb{R}^+$ on the edges. Find a subset $F\subseteq E$ of the edges such that for every $i\in\{1,\dots,k\}$ there is a path between $s_i$ and $t_i$ only using edges in $F$.

$$
\begin{array}{lrccc}
\min & & \sum_e c(e)x_e & & \\
\text{s.t.} & \forall S\subseteq V: S\in S_i \text{ for some } i & \sum_{e\in\delta(S)} x_e & \ge & 1\\
 & \forall e\in E & x_e & \in & \{0,1\}
\end{array}
$$

Here $S_i$ contains all sets $S$ such that $s_i\in S$ and $t_i\notin S$.

$$\begin{array}{llrcl} \max & & \sum_{S\,:\,\exists i \text{ s.t. } S \in S_i} y_S & & \\ \text{s.t.} & \forall e \in E & \sum_{S: e\in\delta(S)} y_S & \le & c(e) \\ & & y_S & \ge & 0 \end{array}$$

The difference to the dual of the shortest path problem is that we have many more variables (sets for which we can generate a moat of non-zero width).



**Algorithm 1** FirstTry

$y \leftarrow 0$
$F \leftarrow \emptyset$
**while** not all $s_i$-$t_i$ pairs connected in $F$ **do**
    Let $C$ be some connected component of $(V, F)$ such that $|C \cap \{s_i, t_i\}| = 1$ for some $i$.
    Increase $y_C$ until there is an edge $e' \in \delta(C)$ s.t. $\sum_{S\in S_i: e'\in\delta(S)} y_S = c_{e'}$
    $F \leftarrow F \cup \{e'\}$
Let $P_i$ be an $s_i$-$t_i$ path in $(V, F)$
**return** $\bigcup_i P_i$



$$\sum_{e\in F} c(e) = \sum_{e\in F} \sum_{S: e\in\delta(S)} y_S = \sum_S |\delta(S) \cap F| \cdot y_S \ .$$

If we show that $y_S > 0$ implies that $|\delta(S) \cap F| \le \alpha$ we are in good shape.

However, this is not true:

- Take a graph on $k+1$ vertices $v_0, v_1, \ldots, v_k$.
- The $i$-th pair is $v_0$-$v_i$.
- The first component $C$ could be $\{v_0\}$.
- We only set $y_{\{v_0\}} = 1$. All other dual variables stay $0$.
- The final set $F$ contains all edges $\{v_0, v_i\}$, $i = 1, \ldots, k$.
- $y_{\{v_0\}} > 0$ but $|\delta(\{v_0\}) \cap F| = k$.



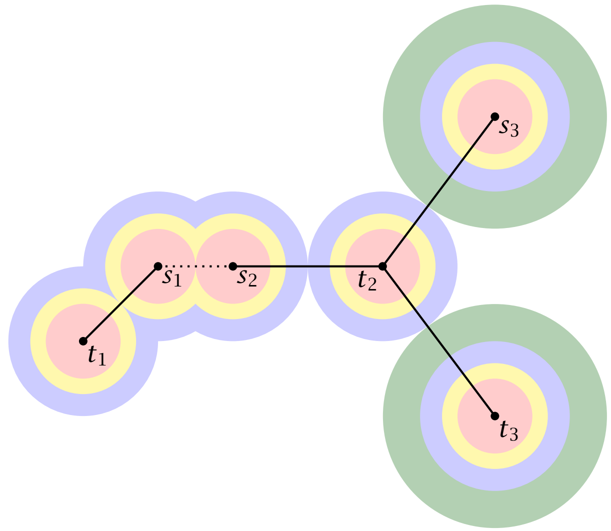
**Algorithm 1** SecondTry

$y \leftarrow 0$; $F \leftarrow \emptyset$; $\ell \leftarrow 0$
**while** not all $s_i$-$t_i$ pairs connected in $F$ **do**
    $\ell \leftarrow \ell + 1$
    Let $\mathcal{C}$ be set of all connected components $C$ of $(V, F)$ such that $|C \cap \{s_i, t_i\}| = 1$ for some $i$.
    Increase $y_C$ for all $C \in \mathcal{C}$ uniformly until for some edge $e_\ell \in \delta(C')$, $C' \in \mathcal{C}$ s.t. $\sum_{S: e_\ell\in\delta(S)} y_S = c_{e_\ell}$
    $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
**for** $k \leftarrow \ell$ downto 1 **do** // reverse deletion
    **if** $F' - e_k$ is feasible solution **then**
        remove $e_k$ from $F'$
**return** $F'$

The reverse deletion step is not strictly necessary this way. It would also be sufficient to simply delete all unnecessary edges in any order.



## Example






**Lemma 17**
*For any $\mathcal{C}$ in any iteration of the algorithm*

$$\sum_{C\in\mathcal{C}} |\delta(C)\cap F'| \leq 2|\mathcal{C}|$$

This means that the number of times a moat from $\mathcal{C}$ is crossed in the final solution is at most twice the number of moats.

**Proof:** later...



$$\sum_{e\in F'} c_e = \sum_{e\in F'} \sum_{S:e\in\delta(S)} y_S = \sum_S |F'\cap\delta(S)|\cdot y_S \ .$$

We want to show that

$$\sum_S |F'\cap\delta(S)|\cdot y_S \leq 2\sum_S y_S$$

- In the $i$-th iteration the increase of the left-hand side is

$$\epsilon \sum_{C\in\mathcal{C}} |F'\cap\delta(C)|$$

  and the increase of the right hand side is $2\epsilon|\mathcal{C}|$.
- Hence, by the previous lemma the inequality holds after the iteration if it holds in the beginning of the iteration.

**Lemma 18**

*For any set of connected components $\mathcal{C}$ in any iteration of the algorithm*

$$\sum_{C\in\mathcal{C}} |\delta(C)\cap F'| \le 2|\mathcal{C}|$$

**Proof:**

- At any point during the algorithm the set of edges forms a forest (why?).
- Fix iteration $i$. $e_i$ is the set we add to $F$. Let $F_i$ be the set of edges in $F$ at the beginning of the iteration.
- Let $H = F' - F_i$.
- All edges in $H$ are necessary for the solution.



- Contract all edges in $F_i$ into single vertices $V'$.
- We can consider the forest $H$ on the set of vertices $V'$.
- Let $\deg(v)$ be the degree of a vertex $v \in V'$ within this forest.
- Color a vertex $v \in V'$ red if it corresponds to a component from $\mathcal{C}$ (an active component). Otw. color it blue. (Let $B$ the set of blue vertices (with non-zero degree) and $R$ the set of red vertices)
- We have

$$\sum_{v\in R}\deg(v) \ge \sum_{C\in\mathcal{C}} |\delta(C)\cap F'| \overset{?}{\le} 2|\mathcal{C}| = 2|R|$$



- Suppose that no node in $B$ has degree one.
- Then

$$\begin{aligned}\sum_{v\in R}\deg(v) &= \sum_{v\in R\cup B}\deg(v) - \sum_{v\in B}\deg(v)\\ &\le 2(|R|+|B|) - 2|B| = 2|R|\end{aligned}$$

- Every blue vertex with non-zero degree must have degree at least two.
  - Suppose not. The single edge connecting $b \in B$ comes from $H$, and, hence, is necessary.
  - But this means that the cluster corresponding to $b$ must separate a source-target pair.
  - But then it must be a red node.