9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- **$\mathcal{P}$. makeset(x):** Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.
- **$\mathcal{P}$. find(x):** Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- **$\mathcal{P}$. union(x, y):** Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

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9 Union Find

Algorithm 1 Kruskal-MST($G = (V, E), w$)

$A \leftarrow \emptyset$;
for all $v \in V$ **do**
    $v.\mathrm{set} \leftarrow \mathcal{P}.\mathrm{makeset}(v.\mathrm{label})$
sort edges in non-decreasing order of weight w
for all $(u, v) \in E$ in non-decreasing order **do**
    if $\mathcal{P}.\mathrm{find}(u.\mathrm{set}) \neq \mathcal{P}.\mathrm{find}(v.\mathrm{set})$ **then**
        $A \leftarrow A \cup \{(u, v)\}$
        $\mathcal{P}.\mathrm{union}(u.\mathrm{set}, v.\mathrm{set})$

List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.

- makeset(x) can be performed in constant time.
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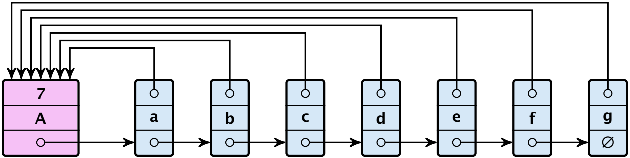

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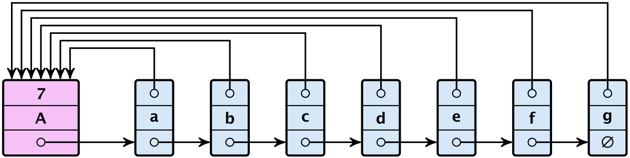

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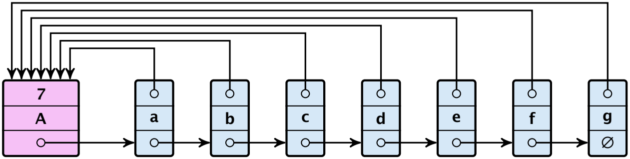

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- Determine sets S_x and S_y.
- Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
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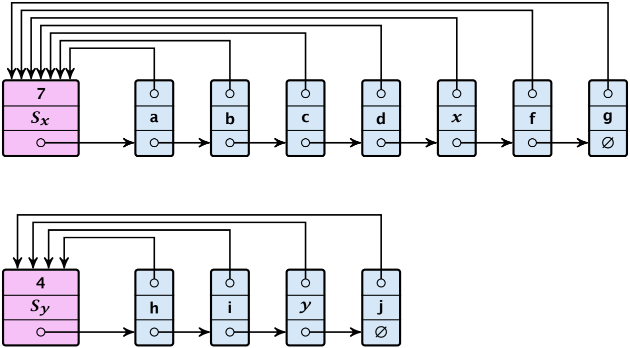

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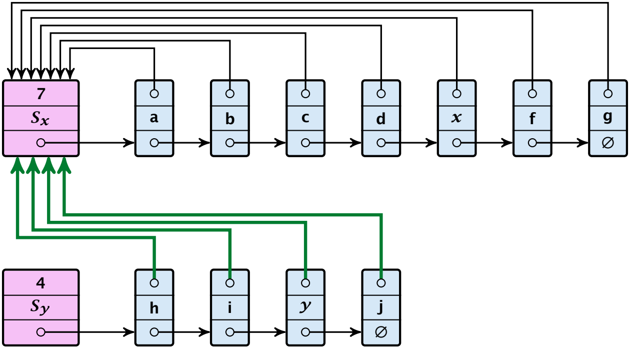

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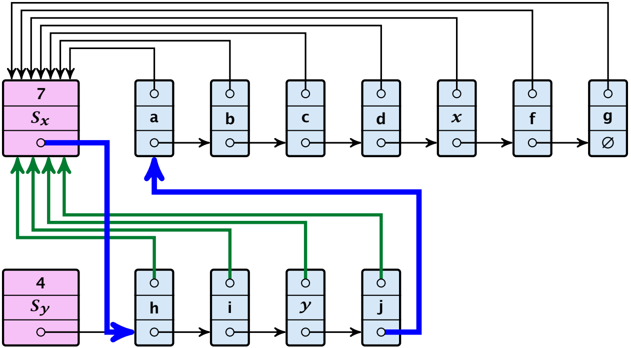

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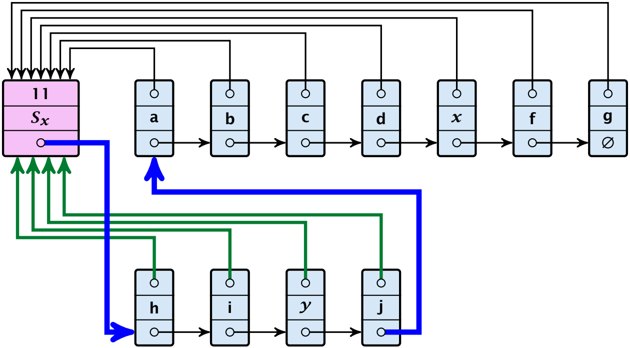

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x, y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 1

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x, y)$: $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.

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makeset(x) : The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

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Lemma 2

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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- Maintain nodes of a set in a tree.
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- Only pointer to parent exists; we cannot list all elements of a given set.
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Set system $\{2, 5, 10, 12\}$, $\{3, 6, 7, 8, 9, 14, 17\}$, $\{16, 19, 23\}$.

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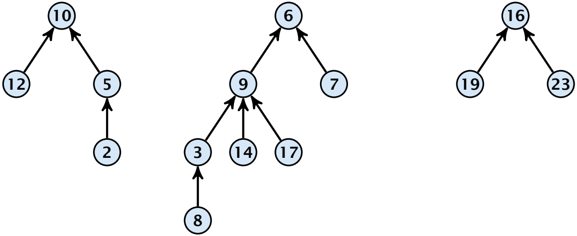

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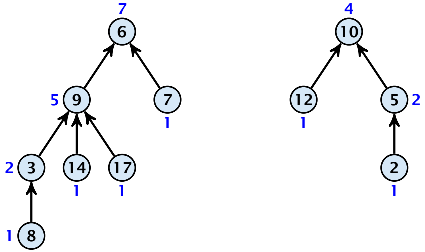

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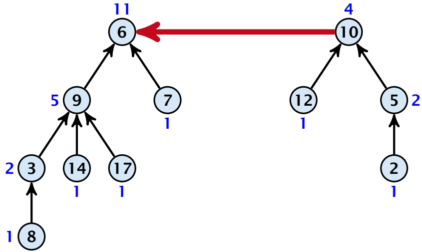

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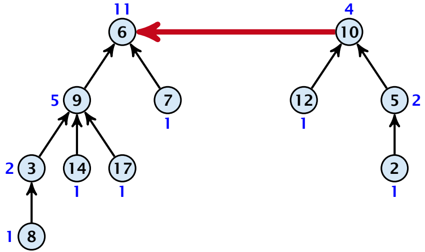

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

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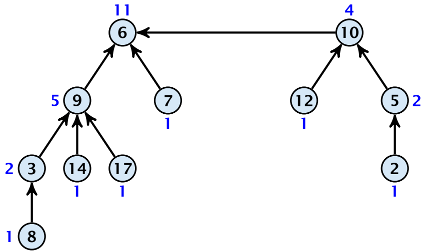

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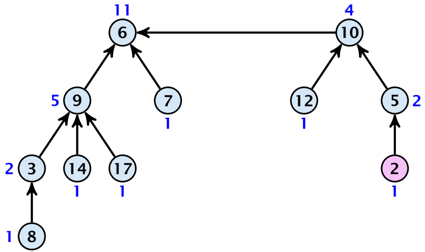

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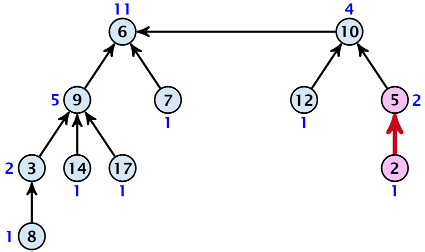

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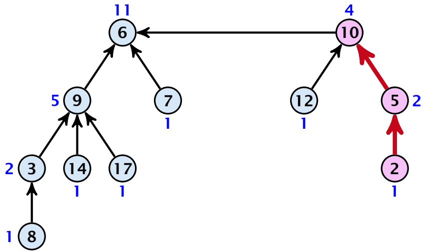

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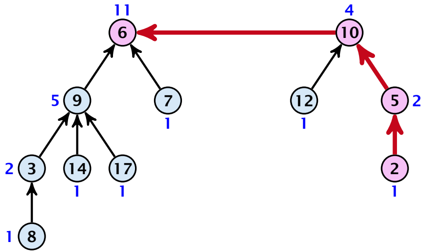

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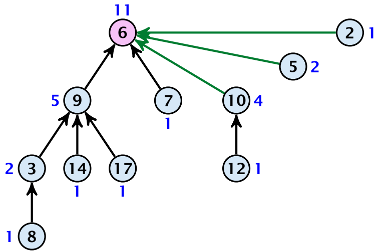

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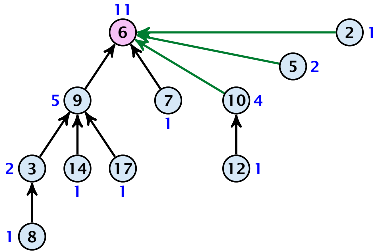

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Amortized Analysis

Definitions:

- $\mathrm{size}(v)$ = the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find operations.

- $\mathrm{rank}(v) = \lfloor \log(\mathrm{size}(v)) \rfloor$.
- $\Longrightarrow \mathrm{size}(v) \geq 2^{\mathrm{rank}(v)}$.

Lemma 4

The rank of a parent must be strictly larger than the rank of a child.

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Lemma 5

There are at most $n/2^s$ nodes of rank s.

Proof.

- Let's say a node v sees node x if v is in x's sub-tree at the time that x becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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Theorem 6

Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

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In the following we assume $n \geq 2$.

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- A node with rank $\text{rank}(v)$ is in rank group $\log^*(\text{rank}(v))$.
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- A rank group $g \geq 1$ contains ranks $\text{tow}(g-1)+1, \ldots, \text{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
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Accounting Scheme:

- create an account for every find-operation
- create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\mathrm{parent}[v]$ as follows:

- If $\mathrm{parent}[v]$ is the root we charge the cost to the find-account.
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Observations:

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Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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Amortized Analysis

The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^{2}}}}_{y+3 \text{ times}} - 3$

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