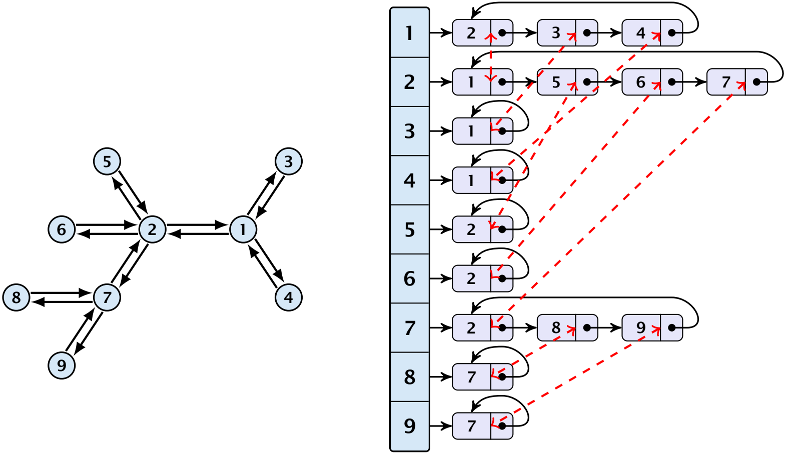
Tree Algorithms



Euler Circuits

Every node v fixes an arbitrary ordering among its adjacent nodes:

$$u_0, u_1, \ldots, u_{d-1}$$

We obtain an Euler tour by setting

$$\mathrm{succ}((u_i, v)) = (v, u_{(i+1) \bmod d})$$

Euler Circuits

Lemma 1
An Euler circuit can be computed in constant time $\mathcal{O}(1)$ with $\mathcal{O}(n)$ operations.

Euler Circuits — Applications

Rooting a tree

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 1$ for every edge;
- perform parallel prefix; let $s[\cdot]$ be the result array
- if $s[(u,v)] < s[(v,u)]$ then u is parent of v;

Euler Circuits — Applications

Postorder Numbering

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 1$ for every edge $(v, \text{parent}(v))$
- assign $x[e] = 0$ for every edge $(\text{parent}(v), v)$
- perform parallel prefix
- $\text{post}(v) = s[(v, \text{parent}(v))]$; $\text{post}(r) = n$

Euler Circuits — Applications

Level of nodes

- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = -1$ for every edge $(v, \mathrm{parent}(v))$
- assign $x[e] = 1$ for every edge $(\mathrm{parent}(v), v)$
- perform parallel prefix
- $\mathrm{level}(v) = s[(\mathrm{parent}(v), v)]$; $\mathrm{level}(r) = 0$

Euler Circuits — Applications

Number of descendants

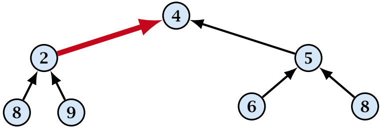
- split the Euler tour at node r
- this gives a list on the set of directed edges (Euler path)
- assign $x[e] = 0$ for every edge $(\text{parent}(v), v)$
- assign $x[e] = 1$ for every edge $(v, \text{parent}(v))$, $v \neq r$
- perform parallel prefix
- $\text{size}(v) = s[(v, \text{parent}(v))] - s[(\text{parent}(v), v)]$

Rake Operation

Given a binary tree T.

Given a leaf $u \in T$ with $p(u) \neq r$ the rake-operation does the following

- remove u and $p(u)$
- attach sibling of u to $p(p(u))$



We want to apply rake operations to a binary tree T until T just consists of the root with two children.

Possible Problems:

1. we could concurrently apply the rake-operation to two siblings
2. we could concurrently apply the rake-operation to two leaves u and v such that $p(u)$ and $p(v)$ are connected

By choosing leaves carefully we ensure that none of the above cases occurs

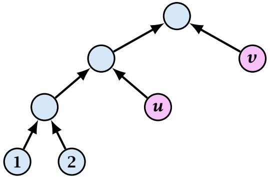
Algorithm:

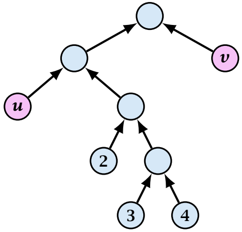
- label leaves consecutively from left to right (excluding left-most and right-most leaf), and store them in an array A
- for $\lceil \log(n+1) \rceil$ iterations
  - apply rake to all odd leaves that are left children
  - apply rake operation to remaining odd leaves (odd at start of round!!!)
  - A=even leaves

Observations

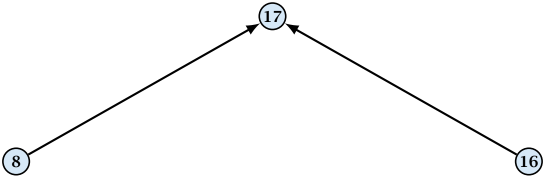
- the rake operation does not change the order of leaves
- two leaves that are siblings do not perform a rake operation in the same round because one is even and one odd at the start of the round
- two leaves that have adjacent parents either have different parity (even/odd) or they differ in the type of child (left/right)

Cases, when the left edge btw. $p(u)$ and $p(v)$ is a left-child edge.





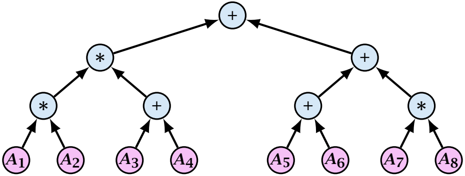
Example

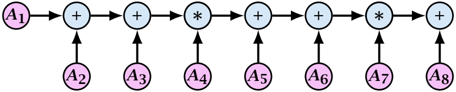


- one iteration can be performed in constant time with $\mathcal{O}(|A|)$ processors, where A is the array of leaves;
- hence, **all** iterations can be performed in $\mathcal{O}(\log n)$ time and $\mathcal{O}(n)$ work;
- the intial parallel prefix also requires time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$

Evaluating Expressions

Suppose that we want to evaluate an expression tree, containing additions and multiplications.





If the tree is not balanced this may be time-consuming.

We can use the rake-operation to do this quickly.

Applying the rake-operation changes the tree.

In order to maintain the value we introduce parameters a_v and b_v for every node that still allows to compute the value of a node based on the value of its children.

Invariant:

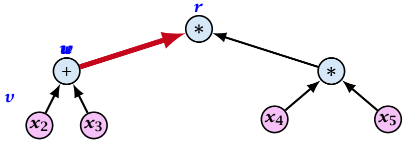
Let u be internal node with children v and w. Then

$$\mathrm{val}(u) = (a_v \cdot \mathrm{val}(v) + b_v) \otimes (a_w \cdot \mathrm{val}(w) + b_w)$$

where $\otimes \in \{*, +\}$ is the operation at node u.

Initially, we can choose $a_v = 1$ and $b_v = 0$ for every node.

Rake Operation



Currently the value at u is

$$\begin{aligned}\mathrm{val}(u) &= (a_v \cdot \mathrm{val}(v) + b_v) + (a_w \cdot \mathrm{val}(w) + b_w) \\ &= x_1 + (a_w \cdot \mathrm{val}(w) + b_w)\end{aligned}$$

In the expression for r this goes in as

$$\begin{aligned}&a_u \cdot [x_1 + (a_w \cdot \mathrm{val}(w) + b_w)] + b_u \\ &\quad = \underbrace{a_u a_w}_{a'_w} \cdot \mathrm{val}(w) + \underbrace{a_u x_1 + a_u b_w + b_u}_{b'_w}\end{aligned}$$

If we change the a and b-values during a rake-operation according to the previous slide we can calculate the value of the root in the end.

Lemma 2

We can evaluate an arithmetic expression tree in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$ regardless of the height or depth of the tree.

By performing the rake-operation in the reverse order we can also compute the value at each node in the tree.

Lemma 3

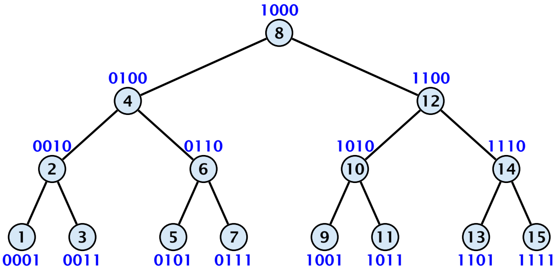
We compute tree functions for arbitrary trees in time $\mathcal{O}(\log n)$ and a linear number of operations.

proof on board...

In the LCA (least common ancestor) problem we are given a tree and the goal is to design a data-structure that answers LCA-queries in constant time.

Least Common Ancestor

LCAs on complete binary trees (inorder numbering):

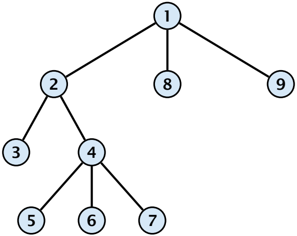


The least common ancestor of u and v is

$$z_1 z_2 \ldots z_i 1 0 \ldots 0$$

where z_{i+1} is the first bit-position in which u and v differ.

Least Common Ancestor



nodes	1	2	3	2	4	5	4	6	4	7	4	2	1	8	1	9	1
levels	0	1	2	1	2	3	2	3	2	3	2	1	0	1	0	1	0

$\ell(v)$ is index of first appearance of v in node-sequence.

$r(v)$ is index of last appearance of v in node-squence.

$\ell(v)$ and $r(v)$ can be computed in constant time, given the node- and level-sequence.

Least Common Ancestor

Lemma 4

1. *u is ancestor of v iff $\ell(u) < \ell(v) < r(u)$*
2. *u and v are not related iff either $r(u) < \ell(v)$ or $\ell(u) < r(v)$*
3. *suppose $r(u) < \ell(v)$ then $\mathrm{LCA}(u, v)$ is vertex with minimum level over interval $[r(u), \ell(v)]$.*

Range Minima Problem

Given an array $A[1 \dots n]$, a range minimum query (ℓ, r) consists of a left index $\ell \in \{1, \dots, n\}$ and a right index $r \in \{1, \dots, n\}$.

The answer has to return the index of the minimum element in the subsequence $A[\ell \dots r]$.

The goal in the range minima problem is to preprocess the array such that range minima queries can be answered quickly (constant time).

Observation

Given an algorithm for solving the range minima problem in time $T(n)$ and work $W(n)$ we can obtain an algorithm that solves the LCA-problem in time $\mathcal{O}(T(n) + \log n)$ and work $\mathcal{O}(n + W(n))$.

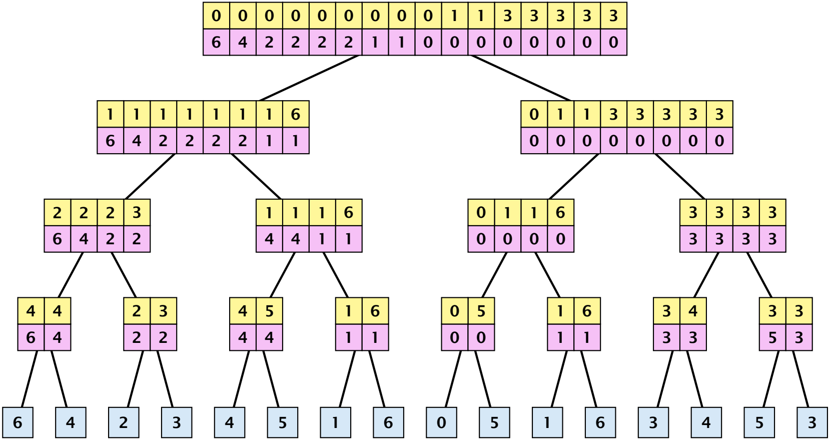
Remark

In the sequential setting the LCA-problem and the range minima problem are equivalent. This is not necessarily true in the parallel setting.

For solving the LCA-problem it is sufficient to solve the restricted range minima problem where two successive elements in the array just differ by $+1$ or -1.

Prefix and Suffix Minima

Tree with prefix-minima and suffix-minima:



- Suppose we have an array A of length $n = 2^k$
- We compute a complete binary tree T with n leaves.
- Each internal node corresponds to a subsequence of A. It contains an array with the prefix and suffix minima of this subsequence.

Given the tree T we can answer a range minimum query (ℓ, r) in constant time.

- we can determine the LCA x of ℓ and r in constant time since T is a complete binary tree
- Then we consider the suffix minimum of ℓ in the left child of x and the prefix minimum of r in the right child of x.
- The minimum of these two values is the result.

Lemma 5

We can solve the range minima problem in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n \log n)$.

Reducing the Work

Partition A into blocks B_i of length $\log n$

Preprocess each B_i block separately by a sequential algorithm so that range-minima queries within the block can be answered in constant time. (**how?**)

For each block B_i compute the minimum x_i and its prefix and suffix minima.

Use the previous algorithm on the array $(x_1, \ldots, x_{n/\log n})$.

Answering a query (ℓ, r):

- if ℓ and r are from the same block the data-structure for this block gives us the result in constant time
- if ℓ and r are from different blocks the result is a minimum of three elements:
  - the suffix minmum of entry ℓ in ℓ's block
  - the minimum among $x_{\ell+1}, \ldots, x_{r-1}$
  - the prefix minimum of entry r in r's block