

Duality

How do we get an upper bound to a maximization LP?

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i -th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

Duality

Definition 2

Let $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$$

is called the **dual problem**.

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

- ▶ $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$
- ▶ $w = -\max\{-b^t y \mid -A^t y \leq -c, y \geq 0\}$

The dual problem is

- ▶ $z = -\min\{-c^t x \mid -Ax \geq -b, x \geq 0\}$
- ▶ $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$

Weak Duality

Let $z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$ and $w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^t y \geq c, y \geq 0\}$.

Theorem 4 (Weak Duality)

Let \hat{x} be primal feasible and let \hat{y} be dual feasible. Then

$$c^t \hat{x} \leq z \leq w \leq b^t \hat{y} .$$

Weak Duality

$$A^t \hat{y} \geq c \Rightarrow \hat{x}^t A^t \hat{y} \geq \hat{x}^t c \quad (\hat{x} \geq 0)$$

$$A \hat{x} \leq b \Rightarrow y^t A \hat{x} \leq y^t b \quad (y \geq 0)$$

This gives

$$c^t \hat{x} \leq y^t A \hat{x} \leq b^t y .$$

Since, there exists primal feasible \hat{x} with $c^t \hat{x} = z$, and dual feasible \hat{y} with $b^t \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

The following linear programs form a primal dual pair:

$$z = \max\{c^t x \mid Ax = b, x \geq 0\}$$

$$w = \min\{b^t y \mid A^t y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\begin{aligned} & \max\{c^t x \mid Ax = b, x \geq 0\} \\ & = \max\{c^t x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ & = \max\{c^t x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

Dual:

$$\begin{aligned} & \min\{[b^t \ -b^t] y \mid [A^t \ -A^t] y \geq c, y \geq 0\} \\ & = \min \left\{ [b^t \ -b^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^t \ -A^t] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\ & = \min \{ b^t \cdot (y^+ - y^-) \mid A^t \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0 \} \\ & = \min \{ b^t y' \mid A^t y' \geq c \} \end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^t - c_B^t A_B^{-1} A \leq 0$$

This is equivalent to $A^t (A_B^{-1})^t c_B \geq c$

$y^* = (A_B^{-1})^t c_B$ is solution to the **dual** $\min\{b^t y \mid A^t y \geq c\}$.

$$\begin{aligned} b^t y^* & = (Ax^*)^t y^* = (A_B x_B^*)^t y^* \\ & = (A_B x_B^*)^t (A_B^{-1})^t c_B = (x_B^*)^t A_B^t (A_B^{-1})^t c_B \\ & = c^t x^* \end{aligned}$$

Hence, the solution is optimal.

Strong Duality

Theorem 5 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D , respectively. Then

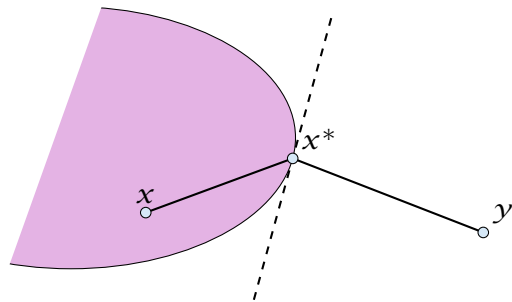
$$z^* = w^*$$

Lemma 6 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X . Then $\min\{f(x) : x \in X\}$ exists.

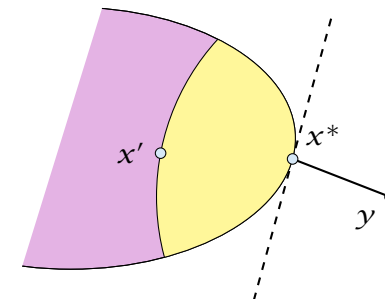
Lemma 7 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y . Moreover for all $x \in X$ we have $(y - x^*)^t(x - x^*) \leq 0$.



Proof of the Projection Lemma

- ▶ Define $f(x) = \|y - x\|$.
- ▶ We want to apply Weierstrass but X may not be bounded.
- ▶ $X \neq \emptyset$. Hence, there exists $x' \in X$.
- ▶ Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned} \|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^t(x - x^*) \end{aligned}$$

Hence, $(y - x^*)^t(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$.

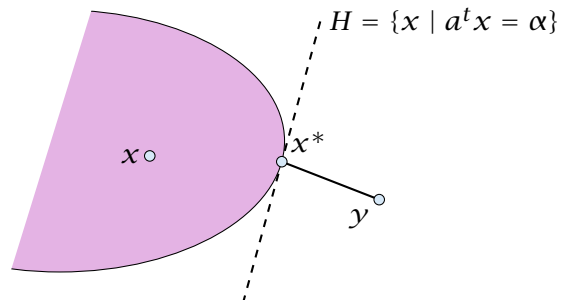
Letting $\epsilon \rightarrow 0$ gives the result.

Theorem 8 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a **separating hyperplane** $\{x \in \mathbb{R}^m : a^t x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that **separates** y from X . ($a^t y < \alpha$; $a^t x \geq \alpha$ for all $x \in X$)

Proof of the Hyperplane Lemma

- ▶ Let $x^* \in X$ be closest point to y in X .
- ▶ By previous lemma $(y - x^*)^t(x - x^*) \leq 0$ for all $x \in X$.
- ▶ Choose $a = (x^* - y)$ and $\alpha = a^t x^*$.
- ▶ For $x \in X$: $a^t(x - x^*) \geq 0$, and, hence, $a^t x \geq \alpha$.
- ▶ Also, $a^t y = a^t(x^* - a) = \alpha - \|a\|^2 < \alpha$



Lemma 9 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then **exactly one** of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^t y \geq 0$, $b^t y < 0$

Assume \hat{x} satisfies 1. and \hat{y} satisfies 2. Then

$$0 > \hat{y}^t b = \hat{y}^t A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^t y \geq 0$, $b^t y < 0$.

Let y be a hyperplane that separates b from S . Hence, $y^t b < \alpha$ and $y^t s \geq \alpha$ for all $s \in S$.

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^t b < 0$$

$y^t Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^t A \geq 0$ as we can choose x arbitrarily large.

Lemma 10 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^t y \geq 0$, $b^t y < 0$, $y \geq 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \geq 0$, $s \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^t \\ I \end{bmatrix} y \geq 0$, $b^t y < 0$

Proof of Strong Duality

$$P: z = \max\{c^t x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^t y \mid A^t y \geq c, y \geq 0\}$$

Theorem 11 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D , respectively (i.e., P and D are non-empty). Then

$$z = w .$$

Proof of Strong Duality

$z \leq w$: follows from weak duality

$z \geq w$:

We show $z < \alpha$ implies $w < \alpha$.

$\begin{array}{l} \exists x \in \mathbb{R}^n \\ \text{s.t. } Ax \leq b \\ \quad -c^t x \leq -\alpha \\ \quad x \geq 0 \end{array}$	$\begin{array}{l} \exists y \in \mathbb{R}^m; v \in \mathbb{R} \\ \text{s.t. } A^t y - cv \geq 0 \\ \quad b^t y - \alpha v < 0 \\ \quad y, v \geq 0 \end{array}$
--	--

From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{array}{l} \exists y \in \mathbb{R}^m; v \in \mathbb{R} \\ \text{s.t. } A^t y - v \geq 0 \\ \quad b^t y - \alpha v < 0 \\ \quad y, v \geq 0 \end{array}$$

If the solution y, v has $v = 0$ we have that

$$\begin{array}{l} \exists y \in \mathbb{R}^m \\ \text{s.t. } A^t y \geq 0 \\ \quad b^t y < 0 \\ \quad y \geq 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^t y < \alpha$. This means that $w < \alpha$.

Fundamental Questions

Definition 12 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^t x \geq \alpha$?

Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? **yes!**
- ▶ Is LP in P?

Proof:

- ▶ Given a primal maximization problem P and a parameter α . Suppose that $\alpha > \text{opt}(P)$.
- ▶ We can prove this by providing an optimal basis for the dual.
- ▶ A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

Complementary Slackness

Lemma 13

Assume a linear program $P = \max\{c^t x \mid Ax \leq b; x \geq 0\}$ has solution x^* and its dual $D = \min\{b^t y \mid A^t y \geq c; y \geq 0\}$ has solution y^* .

1. If $x_j^* > 0$ then the j -th constraint in D is tight.
2. If the j -th constraint in D is not tight then $x_j^* = 0$.
3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^t x^* \leq y^{*t} A x^* \leq b^t y^*$$

Because of strong duality we then get

$$c^t x^* = y^{*t} A x^* = b^t y^*$$

This gives e.g.

$$\sum_j (y^{*t} A - c^t)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^{*t} A \geq c^t$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^{*t} A - c^t)_j > 0$ (the j -th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost
 C, H, M : unit price for corn, hops and malt.

$$\begin{aligned} \min \quad & 480C + 160H + 1190M \\ \text{s.t.} \quad & 5C + 4H + 35M \geq 13 \\ & 15C + 4H + 20M \geq 23 \\ & C, H, M \geq 0 \end{aligned}$$

Note that brewer won't sell (at least not all) if e.g.

$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

Interpretation of Dual Variables

Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and ε_M , respectively.

The profit increases to $\max\{c^t x \mid Ax \leq b + \varepsilon; x \geq 0\}$. Because of strong duality this is equal to

$$\begin{aligned} \min \quad & (b^t + \varepsilon^t) y \\ \text{s.t.} \quad & A^t y \geq c \\ & y \geq 0 \end{aligned}$$

Interpretation of Dual Variables

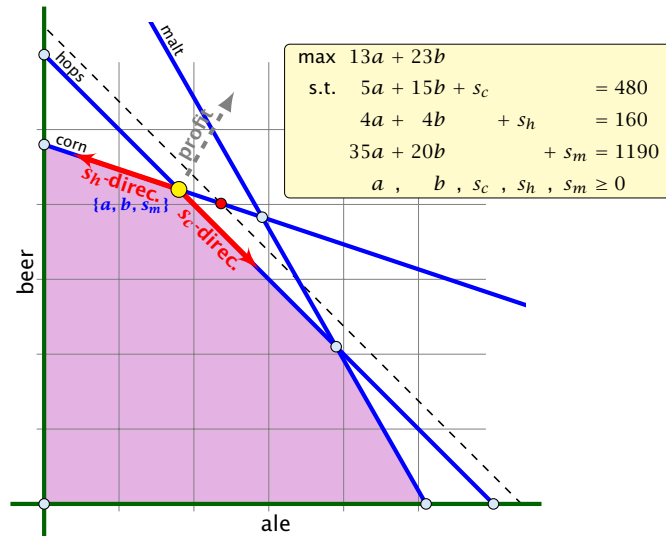
If ε is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

- ▶ If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- ▶ If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

Example



The change in profit when increasing hops by one unit is
 $= \underbrace{c_B^t A_B^{-1}}_{y^*} e_h.$

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 14

An (s, t) -flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f: V \times V \rightarrow \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

(flow conservation constraints)

Flows

Definition 15

The value of an (s, t) -flow f is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an (s, t) -flow with maximum value.

LP-Formulation of Maxflow

$$\begin{array}{ll} \max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t): 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t): 1\ell_{sy} \quad + 1p_y \geq 1 \\ & f_{xs} (x \neq s, t): 1\ell_{xs} - 1p_x \geq -1 \\ & f_{ty} (y \neq s, t): 1\ell_{ty} \quad + 1p_y \geq 0 \\ & f_{xt} (x \neq s, t): 1\ell_{xt} - 1p_x \geq 0 \\ & f_{st}: 1\ell_{st} \geq 1 \\ & f_{ts}: 1\ell_{ts} \geq -1 \\ & \ell_{xy} \geq 0 \end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t): 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t): 1\ell_{sy} - 1 + 1p_y \geq 0 \\ & f_{xs} (x \neq s, t): 1\ell_{xs} - 1p_x + 1 \geq 0 \\ & f_{ty} (y \neq s, t): 1\ell_{ty} - 0 + 1p_y \geq 0 \\ & f_{xt} (x \neq s, t): 1\ell_{xt} - 1p_x + 0 \geq 0 \\ & f_{st}: 1\ell_{st} - 1 + 0 \geq 0 \\ & f_{ts}: 1\ell_{ts} - 0 + 1 \geq 0 \\ & \ell_{xy} \geq 0 \end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t): 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t): 1\ell_{sy} - p_s + 1p_y \geq 0 \\ & f_{xs} (x \neq s, t): 1\ell_{xs} - 1p_x + p_s \geq 0 \\ & f_{ty} (y \neq s, t): 1\ell_{ty} - p_t + 1p_y \geq 0 \\ & f_{xt} (x \neq s, t): 1\ell_{xt} - 1p_x + p_t \geq 0 \\ & f_{st}: 1\ell_{st} - p_s + p_t \geq 0 \\ & f_{ts}: 1\ell_{ts} - p_t + p_s \geq 0 \\ & \ell_{xy} \geq 0 \end{array}$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy}: 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & \ell_{xy} \geq 0 \\ & p_s = 1 \\ & p_t = 0 \end{array}$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.