# Parallel Comparison Tree Model

A parallel comparison tree (with parallelism $p$) is a $3^p$-ary tree.

- each internal node represents a set of $p$ comparisons btw. $p$ pairs (not necessarily distinct)
- a leaf $v$ corresponds to a unique permutation that is valid for all the comparisons on the path from the root to $v$
- the number of parallel steps is the height of the tree

# Comparison PRAM

A comparison PRAM is a PRAM where we can only compare the input elements;

- we cannot view them as strings
- we cannot do calculations on them

A lower bound for the comparison tree with parallelism $p$ directly carries over to the comparison PRAM with $p$ processors.

# A Lower Bound for Searching

**Theorem 1**
*Given a sorted table $X$ of $n$ elements and an element $y$. Searching for $y$ in $X$ requires $\Omega(\frac{\log n}{\log(p+1)})$ steps in the parallel comparsion tree with parallelism $p < n$.*

# A Lower Bound for Maximum

**Theorem 2**

*A graph $G$ with $m$ edges and $n$ vertices has an independent set on at least $\frac{n^2}{2m+n}$ vertices.*

**base case ($n = 1$)**

- The only graph with one vertex has $m = 0$, and an independent set of size 1.

**induction step ($1, \dots, n \to n+1$)**

- Let $G$ be a graph with $n+1$ vertices, and $v$ a node with minimum degree ($d$).
- Let $G'$ be the graph after deleting $v$ and its adjacent vertices in $G$.
- $n' = n - (d+1)$
- $m' \le m - \frac{d}{2}(d+1)$ as we remove $d+1$ vertices, each with degree at least $d$
- In $G'$ there is an independent set of size $((n')^2/(2m'+n'))$.
- By adding $v$ we obtain an indepent set of size

$$1 + \frac{(n')^2}{2m'+n'} \ge \frac{n^2}{2m+n}$$

# A Lower Bound for Maximum

**Theorem 3**

*Computing the maximum of $n$ elements in the comparison tree requires $\Omega(\log\log n)$ steps whenever the degree of parallelism is $p \le n$.*

**Theorem 4**

*Computing the maximum of $n$ elements requires $\Omega(\log\log n)$ steps on the comparison PRAM with $n$ processors.*

An adversary can specify the input such that at the end of the $(i+1)$-st step the maximum lies in a set $C_{i+1}$ of size $s_{i+1}$ such that

- no two elements of $C_{i+1}$ have been compared
- $s_{i+1} \geq \frac{s_i^2}{2p+c_i}$

**Theorem 5**

*The selection problem requires* $\Omega(\log n / \log\log n)$ *steps on a comparison PRAM.*

not proven yet

# A Lower Bound for Merging

The $(k, s)$-merging problem, asks to merge $k$ pairs of subsequences $A^1, \dots, A^k$ and $B^1, \dots, B^k$ where we know that all elements in $A^i \cup B^i$ are smaller than elements in $A^j \cup B^j$ for $(i < j)$. Further $|A_i|, |B_i| \geq s$.

# A Lower Bound for Merging

**Lemma 6**

*Suppose we are given a parallel comparison tree with parallelism $p$ to solve the $(k, s)$ merging problem. After the first step an adversary can specify the input such that an arbitrary $(k', s')$ merging problem has to be solved, where*

$$k' = \frac{3}{4}\sqrt{pk}$$

$$s' = \frac{s}{4}\sqrt{\frac{k}{p}}$$

# A Lower Bound for Merging

Partition $A^i s$ and $B^i s$ into blocks of length roughly $s/\ell$; hence $\ell$ blocks.

Define an $\ell \times \ell$ binary matrix $M^i$, where $M^i_{xy}$ is $0$ iff the parallel step **did not** compare an element from $A^i_x$ with an element from $B^i_y$.

The matrix has $2\ell - 1$ diagonals.

Choose for every $i$ the diagonal of $M^i$ that has most zeros.

Pair all $A^i_{j+d_i}, B^i_j$, (where $d_i \in \{-(\ell-1), \dots, \ell-1\}$ specifies the chosen diagonal) for which the entry in $M^i$ is zero.

We can choose value s.t. elements for the $j$-th pair along the diagonal are **all** smaller than for the $(j+1)$-th pair.

Hence, we get a $(k', s')$ problem.

## How many pairs do we have?

- there are $k\ell$ blocks in total
- there are $k \cdot \ell^2$ matrix entries in total
- there are at least $k \cdot \ell^2 - p$ zeros.
- choosing a random diagonal (same for every matrix $M^i$) hits at least

$$\frac{k\ell^2 - p}{2\ell - 1} \geq \frac{k\ell}{2} - \frac{p}{2\ell}$$

zeroes.
- Choosing $\ell = \lceil 2\sqrt{p/k} \rceil$ gives

$$k' \geq \frac{3}{4}\sqrt{pk} \text{ and } s' = \lfloor \frac{s}{\ell} \rfloor \geq \frac{s}{4\sqrt{p/k}} = \frac{s}{4}\sqrt{\frac{k}{p}}$$

where we assume $s \geq 6\sqrt{p/k}$.

## Lemma 7

*Let $T(k, s, p)$ be the number of parallel steps required on a comparison tree to solve the $(k, s)$ merging problem. Then*

$$T(k, p, s) \geq \frac{1}{4} \log \frac{\log \frac{p}{k}}{\log \frac{p}{ks}}$$

*provided that $p \geq 2ks$ and $p \leq ks^2/36$*

**Induction Step:**

Assume that

$$
\begin{aligned}
T(k',s',p) &\geq \frac{1}{4}\log\frac{\log\frac{p}{k'}}{\log\frac{p}{k's'}} \\
&\geq \frac{1}{4}\log\frac{\log\frac{4}{3}\sqrt{\frac{p}{k}}}{\log\frac{16}{3}\frac{p}{ks}} \\
&\geq \frac{1}{4}\log\frac{\frac{1}{2}\log\frac{p}{k}}{7\log\frac{p}{ks}} \\
&\geq \frac{1}{4}\log\frac{\log\frac{p}{k}}{\log\frac{p}{ks}} - 1
\end{aligned}
$$

This gives the induction step.

**Theorem 8**

*Merging requires at least $\Omega(\log\log n)$ time on a CRCW PRAM with $n$ processors.*