4.6 Symmetry Breaking

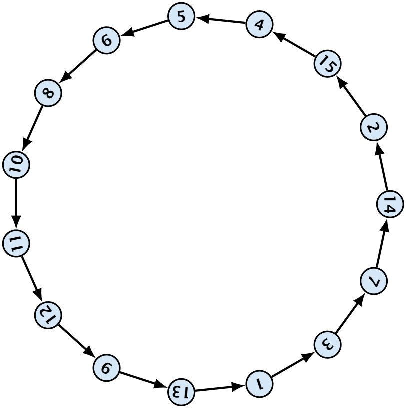
The following algorithm colors an n-node cycle with $\lceil \log n \rceil$ colors.

Algorithm 9 BasicColoring

for $1 \le i \le n$ **pardo**
  $\mathrm{col}(i) \leftarrow i$
  $k_i \leftarrow$ smallest bitpos where $\mathrm{col}(i)$ and $\mathrm{col}(S(i))$ differ
  $\mathrm{col}'(i) \leftarrow 2k_i + \mathrm{col}(i)_{k_i}$

(bit positions are numbered starting with 0)

4.6 Symmetry Breaking



v	col	k	col′
1	0001	1	2
3	0011	2	4
7	0111	0	1
14	1110	2	5
2	0010	0	0
15	1111	0	1
4	0100	0	0
5	0101	0	1
6	0110	1	3
8	1000	1	2
10	1010	0	0
11	1011	0	1
12	1100	0	0
9	1001	2	4
13	1101	2	5

4.6 Symmetry Breaking

Applying the algorithm to a coloring with bit-length t generates a coloring with largest color at most

$$2(t-1)+1$$

and bit-length at most

$$\lceil\log_2(2(t-1)+1)\rceil \le \lceil\log_2(2t)\rceil = \lceil\log_2(t)\rceil + 1$$

Applying the algorithm repeatedly generates a constant number of colors after $\mathcal{O}(\log^* n)$ operations.

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4.6 Symmetry Breaking

As long as the bit-length $t \geq 4$ the bit-length decreases.

Applying the algorithm with bit-length 3 gives a coloring with colors in the range $0, \ldots, 5 = 2t - 1$.

We can improve to a 3-coloring by successively re-coloring nodes from a color-class:

Algorithm 10 ReColor

```
for ℓ ← 5 to 3
    for 1 ≤ i ≤ n pardo
        if col(i) = ℓ then
            col(i) ← min{{0,1,2} \ {col(P[i]), col(S[i])}}
```

This requires time $\mathcal{O}(1)$ and work $\mathcal{O}(n)$.

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4.6 Symmetry Breaking

Lemma 7

We can color vertices in a ring with three colors in $\mathcal{O}(\log^* n)$ *time and with* $\mathcal{O}(n \log^* n)$ *work.*

not work optimal

4.6 Symmetry Breaking

Lemma 8

Given n integers in the range $0, \ldots, \mathcal{O}(\log n)$, there is an algorithm that sorts these numbers in $\mathcal{O}(\log n)$ time using a linear number of operations.

Proof: Exercise!

4.6 Symmetry Breaking

Algorithm 11 OptColor

for $1 \leq i \leq n$ **pardo**
    $\mathrm{col}(i) \leftarrow i$
apply BasicColoring once
sort vertices by colors
for $\ell = 2\lceil \log n \rceil$ **to** 3 **do**
    for all vertices i of color ℓ **pardo**
        $\mathrm{col}(i) \leftarrow \min\{\{0,1,2\} \setminus \{\mathrm{col}(P[i]), \mathrm{col}(S[i])\}\}$

Lemma 9

A ring can be colored with 3 *colors in time* $\mathcal{O}(\log n)$ *and with work* $\mathcal{O}(n)$.

work optimal but not too fast