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Fall Semester
December 22, 2014

Randomized Algorithms

Exercise Sheet 10

Due: January 12, 2015

Exercise 1a: (10 points)
Let $X_1, X_2, \ldots, X_n$ be independent random variables such that X_i is equal to 1 with probability p_i and equal to 0 with probability $1 - p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$. For any $\delta > 1$, show that

$$Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta \ln \delta}{2}\mu}$$

Exercise 1b: (10 points)
In the Cyclic Shift problem, given a $n \times n$ matrix A with entries in $\{0, 1\}$, the objective is to shift the rows cyclically so that the maximum column sum is minimized. For example, consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

which has maximum column sum 3. By shifting the second row by one place we get the matrix

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

with maximum column sum 2. Consider the following randomized algorithm for the Cyclic Shift problem:

- for each $i = 1, 2, \ldots, n$
  choose $s_i \in \{0, 1, \ldots, n-1\}$ uniformly at random
  shift the i-th row by s_i places

Let $c = \sum_{i=1}^n \frac{c_i}{n}$, where c_i is the number of entries in the i-th row which are equal to 1. We say that the input matrix A is sparse if $c \leq \ln n$. Consider a column in the matrix produced by the algorithm and let X be a random variable which corresponds to the sum of the elements in that column. Show that

- $Pr(X \geq 5c) \leq \frac{1}{n^2}$ when A is not sparse, and
- $Pr(X > c + 4 \ln n) \leq \frac{1}{n^2}$ when A is sparse.

Exercise 2a: (10 points)
Consider the bit-fixing algorithm for permutation routing on the n-cube where n is even. Recall that, in a permutation routing, each node sends exactly one packet and receives exactly one packet. Consider the permutation in which, if a source node s is the concatenation of two binary strings a_s and b_s each of length $\frac{n}{2}$, then the destination of the packet sent from s is the concatenation of b_s and a_s. Show that, given this permutation, the bit-fixing algorithm takes $\Omega(\sqrt{N})$ steps, where $N = 2^n$ is the number of nodes of the n-cube.

Exercise 2b: (10 points)
Consider the following variant of the bit-fixing algorithm for permutation routing on the n-cube. Instead of fixing the bits in order from 1 to n, each packet chooses a random order (independently from the other packets) and fixes the bits in that order. Show that there is a permutation such that the expected number of steps taken by this algorithm is $2^{\Omega(n)}$.

Hint: Consider the same permutation with the previous exercise. That is, each packet has a source node $(a_1, a_2, \ldots, a_{n/2}, b_1, b_2, \ldots, b_{n/2})$ *and a destination node* $(b_1, b_2, \ldots, b_{n/2}, a_1, a_2, \ldots, a_{n/2})$*. Then, show that the expected number of packets that pass through the node* $(0, 0, \ldots, 0)$ *is* $2^{\Omega(n)}$*. You may use the inequality* $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$.