Prof. Dr. Susanne Albers
Dr. Dimitrios Letsios
Lehrstuhl Theoretische Informatik
Fakultät für Informatik
Technische Universität München

Fall Semester
December 1, 2014

Randomized Algorithms

Exercise Sheet 7

Due: December 8, 2014

Exercise 1: (10 points)
Consider an arbitrary function $f : A_n \to \mathbb{N}$, where $A_n = \{1, 2, \ldots, n\}$ and $\mathbb{N}$ is the set of natural numbers. We know nothing about f, but we are given a black-box procedure which returns $f(x)$ for any input value $x \in A_n$. The median of f is any $M \in \mathbb{N}$ such that at least half elements $x \in A_n$ satisfy $f(x) \geq M$ and at least half elements $x \in A_n$ satisfy $f(x) \leq M$. Our objective is to compute two elements $a, b \in A_n$ such that $f(a) \geq M$ and $f(b) \leq M$.

- Propose a zero-error algorithm for computing a and b in the case where we know M.
- Propose an algorithm which computes a and b with error probability at most q for the case in which we do not know M.

The expected running times of the two algorithms must be as low as possible. Assume that we are given an efficient procedure for choosing an element of A_n uniformly at random.

Exercise 2: (10 points)
Recall the randomized median algorithm that we presented in class for finding the median of a set S with n elements. We showed that the algorithm runs in $O(n)$ time and it might fail with probability $O(\frac{1}{n^{1/4}})$. What is the number of random bits required by the algorithm? Modify the algorithm so that it uses only $O(\log n)$ random bits and it fails with probability $O(\frac{1}{n})$. Assume that n is a prime.

Hint: Use the idea of two-point sampling.

Exercise 3: (10 points)
Consider a different version of the stable marriage problem in which men and women might be indifferent between certain options. There are two sets M and W with n men and n women, respectively. Each person has a preference list of the members of the opposite sex but there might be ties. For example, a woman w might prefer man m to man m' or she might be indifferent between them. Each preference list is organized in non-increasing order of desirability. Now, there are two different kinds of instabilities that might occur in a marriage.

- Strong Instability: There exists a man m and a woman w such that they both prefer each other to their actual partners. Can we always find a marriage without strong instabilities?

- Weak Instability: There exists a man m and a woman w married with a woman w' and a man m', respectively, such that one of the following holds:
  - m prefers w to w' and w prefers m to m' or she is indifferent between them, or
  - w prefers m to m' and m prefers w to w' or he is indifferent between them.

  Can we always find a marriage without weak instabilities?

Exercise 4: (10 points)
Recall that the Coupon Collector's problem can be viewed as a balls-into-bins problem. In class, we showed that if we throw $n \ln n + cn$ balls into n bins, where c is a constant, then the probability that a specific bin is empty is approximately $\frac{1}{ne^c}$. Show that the probability that some bin is empty is at least $\frac{1}{e^c} - \frac{1}{2e^{2c}}$, for large n.

Hint: By the inclusion-exclusion principle, for any events $E_1, E_2, \ldots, E_n$

$$Pr(E_1 \cup E_2 \cup \ldots \cup E_n) \geq \sum_i Pr(E_i) - \sum_{i \neq j} Pr(E_i \cap E_j)$$

Moreover, for large n

$$\left(1 - \frac{c}{n}\right)^n = e^{-c}$$