## Duality

**How do we get an upper bound to a maximization LP?**

$$\begin{array}{rrcrl} \max & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a,b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of $800$).

If you take a conic combination of the rows (multiply the $i$-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.



## Duality

**Definition 2**
Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ be a linear program $P$ (called the primal linear program).

The linear program $D$ defined by

$$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$$

is called the dual problem.



## Duality

**Lemma 3**
*The dual of the dual problem is the primal problem.*

**Proof:**

- $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$
- $w = -\max\{-b^T y \mid -A^T y \le -c, y \ge 0\}$

The dual problem is

- $z = -\min\{-c^T x \mid -Ax \ge -b, x \ge 0\}$
- $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$



## Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$ be a primal dual pair.

$x$ is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

$y$ is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

**Theorem 4 (Weak Duality)**
*Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then*

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

## Weak Duality

$A^T\hat{y} \ge c \Rightarrow \hat{x}^T A^T \hat{y} \ge \hat{x}^T c \; (\hat{x} \ge 0)$

$A\hat{x} \le b \Rightarrow y^T A\hat{x} \le \hat{y}^T b \; (\hat{y} \ge 0)$

This gives

$$c^T\hat{x} \le \hat{y}^T A\hat{x} \le b^T\hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^Ty = w$ we get $z \le w$.

If $P$ is unbounded then $D$ is infeasible.



The following linear programs form a primal dual pair:

$$z = \max\{c^Tx \mid Ax = b, x \ge 0\}$$
$$w = \min\{b^Ty \mid A^Ty \ge c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.



## 5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^Tx \mid Ax = b, x \ge 0\}$$
$$w = \min\{b^Ty \mid A^Ty \ge c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.



## Proof

**Primal:**

$$\begin{aligned}
&\max\{c^Tx \mid Ax = b, x \ge 0\}\\
&\quad = \max\{c^Tx \mid Ax \le b, -Ax \le -b, x \ge 0\}\\
&\quad = \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \le \begin{bmatrix} b \\ -b \end{bmatrix}, x \ge 0\}
\end{aligned}$$

**Dual:**

$$\begin{aligned}
&\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\}\\
&\quad = \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\}\\
&\quad = \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \ge c, y^- \ge 0, y^+ \ge 0\right\}\\
&\quad = \min\left\{b^Ty' \;\middle|\; A^Ty' \ge c\right\}
\end{aligned}$$

## Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

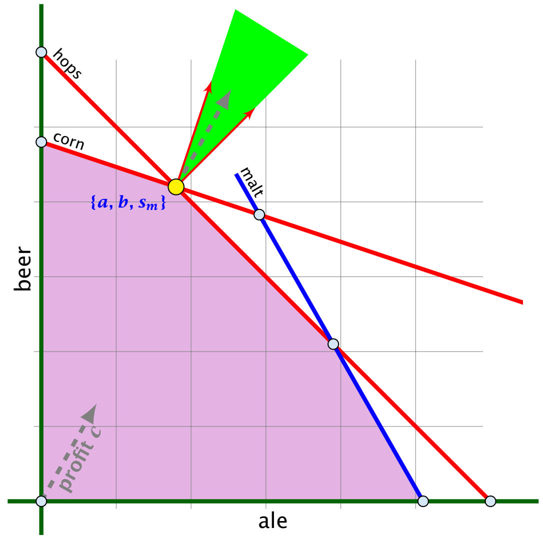
$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

Hence, the solution is optimal.



## ?




The profit vector $c$ lies in the cone generated by the normals for the hops and the corn constraint.

## Strong Duality

**Theorem 5 (Strong Duality)**

*Let $P$ and $D$ be a primal dual pair of linear programs, and let $z^*$ and $w^*$ denote the optimal solution to $P$ and $D$, respectively. Then*

$$z^* = w^*$$



## Strong Duality

**Theorem 6 (Strong Duality)**

*Let $P$ and $D$ be a primal dual pair of linear programs, and let $z^*$ and $w^*$ denote the optimal solution to $P$ and $D$, respectively. Then*

$$z^* = w^*$$

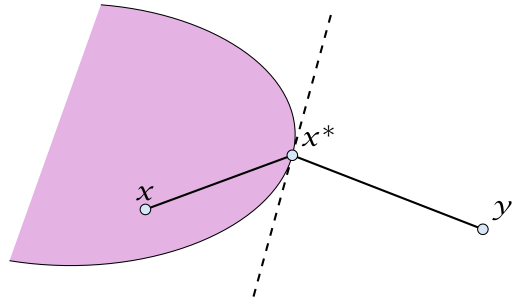
**Lemma 7 (Weierstrass)**

*Let $X$ be a compact set and let $f(x)$ be a continuous function on $X$. Then $\min\{f(x) : x \in X\}$ exists.*



**Lemma 8 (Projection Lemma)**

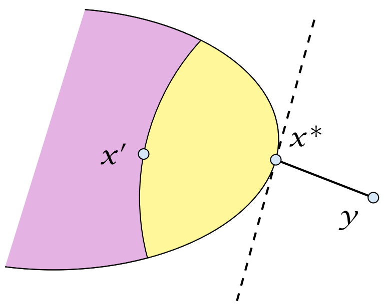
*Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from $y$. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.*






# Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but $X$ may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.






# Proof of the Projection Lemma (continued)

$x^*$ is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2\\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

Hence, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon\|x - x^*\|^2$.

Letting $\epsilon \to 0$ gives the result.

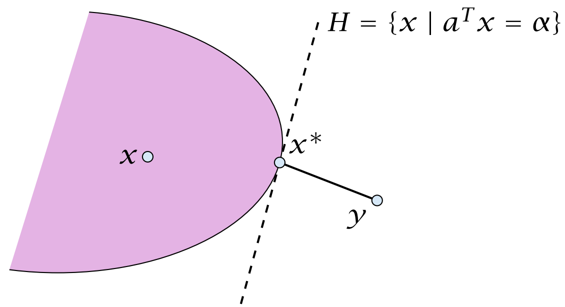
**Theorem 9 (Separating Hyperplane)**

*Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates $y$ from $X$. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)*



## Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to $y$ in $X$.
- By previous lemma $(y - x^*)^T (x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
- For $x \in X : a^T (x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
- Also, $a^T y = a^T (x^* - a) = \alpha - \|a\|^2 < \alpha$






**Lemma 10 (Farkas Lemma)**

*Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.*

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.



## Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that $S$ closed, convex, $b \notin S$.

We want to show that there is $y$ with $A^T y \geq 0$, $b^T y < 0$.

Let $y$ be a hyperplane that separates $b$ from $S$. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T A x \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose $x$ arbitrarily large.

**Lemma 11 (Farkas Lemma; different version)**

*Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.*

1. $\exists x \in \mathbb{R}^n$ *with* $Ax \le b$, $x \ge 0$
2. $\exists y \in \mathbb{R}^m$ *with* $A^T y \ge 0$, $b^T y < 0$, $y \ge 0$

**Rewrite the conditions:**

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
2. $\exists y \in \mathbb{R}^m$ with $\begin{bmatrix} A^T \\ I \end{bmatrix} y \ge 0$, $b^T y < 0$



## Proof of Strong Duality

$P$: $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

$D$: $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$

**Theorem 12 (Strong Duality)**

*Let $P$ and $D$ be a primal dual pair of linear programs, and let $z$ and $w$ denote the optimal solution to $P$ and $D$, respectively (i.e., $P$ and $D$ are non-empty). Then*

$$z = w \ .$$



## Proof of Strong Duality

$z \le w$: follows from weak duality

$z \ge w$:
We show $z < \alpha$ implies $w < \alpha$.

$$\begin{array}{lrcl} \exists x \in \mathbb{R}^n & & & \\ \text{s.t.} & Ax & \le & b \\ & -c^T x & \le & -\alpha \\ & x & \ge & 0 \end{array}$$

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^T y - cv & \ge & 0 \\ & b^T y - \alpha v & < & 0 \\ & y, v & \ge & 0 \end{array}$$

From the definition of $\alpha$ we know that the first system is infeasible; hence the second must be feasible.



## Proof of Strong Duality

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m; v \in \mathbb{R} & & & \\ \text{s.t.} & A^T y - v & \ge & 0 \\ & b^T y - \alpha v & < & 0 \\ & y, v & \ge & 0 \end{array}$$

If the solution $y, v$ has $v = 0$ we have that

$$\begin{array}{lrcl} \exists y \in \mathbb{R}^m & & & \\ \text{s.t.} & A^T y & \ge & 0 \\ & b^T y & < & 0 \\ & y & \ge & 0 \end{array}$$

is feasible. By Farkas lemma this gives that LP $P$ is infeasible. Contradiction to the assumption of the lemma.

## Proof of Strong Duality

Hence, there exists a solution $y, v$ with $v > 0$.

We can rescale this solution (scaling both $y$ and $v$) s.t. $v = 1$.

Then $y$ is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.



## Fundamental Questions

**Definition 13 (Linear Programming Problem (LP))**

Let $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

**Questions**:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

**Proof**:

- Given a primal maximization problem $P$ and a parameter $\alpha$. Suppose that $\alpha > \mathrm{opt}(P)$.
- We can prove this by providing an optimal basis for the dual.
- A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.



## Complementary Slackness

**Lemma 14**

*Assume a linear program $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$ has solution $x^*$ and its dual $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$ has solution $y^*$.*

1. *If $x_j^* > 0$ then the $j$-th constraint in $D$ is tight.*
2. *If the $j$-th constraint in $D$ is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the $i$-th constraint in $P$ is tight.*
4. *If the $i$-th constraint in $P$ is not tight than $y_i^* = 0$.*

If we say that a variable $x_j^*$ ($y_i^*$) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.



## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \geq c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the $j$-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

## Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$
\begin{array}{rrcrl}
\text{max} & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \le 480 \\
 & 4a & + & 4b & \le 160 \\
 & 35a & + & 20b & \le 1190 \\
 & & & a, b & \ge 0
\end{array}
$$

- Entrepeneur: buy resources from brewer at minimum cost
  $C$, $H$, $M$: unit price for corn, hops and malt.

$$
\begin{array}{rrcrcrl}
\text{min} & 480C & + & 160H & + & 1190M & \\
\text{s.t.} & 5C & + & 4H & + & 35M & \ge 13 \\
 & 15C & + & 4H & + & 20M & \ge 23 \\
 & & & & & C, H, M & \ge 0
\end{array}
$$

Note that brewer won't sell (at least not all) if e.g.
$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

## Interpretation of Dual Variables

**Marginal Price:**

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and $\varepsilon_M$, respectively.

The profit increases to $\max\{c^T x \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$
\begin{array}{lrcl}
\min & (b^T + \epsilon^T)y & & \\
\text{s.t.} & A^T y & \ge & c \\
 & y & \ge & 0
\end{array}
$$



## Interpretation of Dual Variables

If $\epsilon$ is "small" enough then the optimum dual solution $y^*$ might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

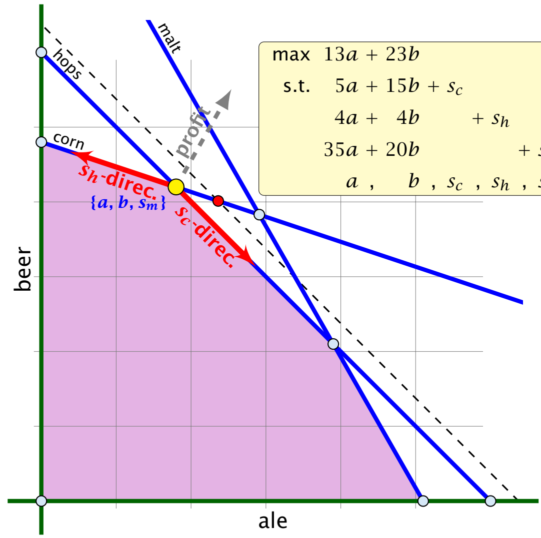
Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.



## Example




$$
\begin{array}{lrrrrrl}
\text{max} & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
 & 4a + 4b & & + s_h & & = & 160 \\
 & 35a + 20b & & & + s_m & = & 1190 \\
 & a\ ,\ b & ,\ s_c & ,\ s_h & ,\ s_m & \ge & 0
\end{array}
$$

The change in profit when increasing hops by one unit is
$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h$.

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.



## Flows

**Definition 15**
An $(s,t)$-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge $(x,y)$

$$0 \leq f_{xy} \leq c_{xy} \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$

(flow conservation constraints)



## Flows

**Definition 16**
The value of an $(s,t)$-flow $f$ is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

**Maximum Flow Problem:**
Find an $(s,t)$-flow with maximum value.



## LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \leq & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \geq & 0 &
\end{array}$$

$$\begin{array}{lllcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} \qquad\quad + 1p_y & \geq & 1 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \geq & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} \qquad\quad + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \geq & 0 \\
 & f_{st}: & 1\ell_{st} & \geq & 1 \\
 & f_{ts}: & 1\ell_{ts} & \geq & -1 \\
 & & \ell_{xy} & \geq & 0
\end{array}$$

## LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - \ \ 1 + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + \ \ 1 & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - \ \ 0 + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + \ \ 0 & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - \ \ 1 + \ \ 0 & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - \ \ 0 + \ \ 1 & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$



## LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - \ \ p_s + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + \ \ p_s & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - \ \ p_t + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + \ \ p_t & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - \ \ p_s + \ \ p_t & \geq & 0 \\
 & f_{ts}: & 1\ell_{ts} - \ \ p_t + \ \ p_s & \geq & 0 \\
 & & \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.



## LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & & \ell_{xy} & \geq & 0 \\
 & & p_s & = & 1 \\
 & & p_t & = & 0
\end{array}
$$

We can interpret the $\ell_{xy}$ value as assigning a length to every edge.

The value $p_x$ for a variable, then can be seen as the distance of $x$ to $t$ (where the distance from $s$ to $t$ is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).



One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value $1$ on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

## Flows

### Definition 17

An $(s,t)$-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge $(x,y)$
$$0 \le f_{xy} \le c_{xy} \ .$$
(capacity constraints)
2. For each $v \in V \setminus \{s,t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$
(flow conservation constraints)



## Flows

### Definition 18

The value of an $(s,t)$-flow $f$ is defined as
$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

**Maximum Flow Problem:**
Find an $(s,t)$-flow with maximum value.



## LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
& \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& & f_{zw} & \ge & 0 &
\end{array}$$

$$\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
& f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
& f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
& f_{ty}\ (y \ne s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
& f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
& f_{st}: & 1\ell_{st} & \ge & 1 \\
& f_{ts}: & 1\ell_{ts} & \ge & -1 \\
& & \ell_{xy} & \ge & 0
\end{array}$$



## LP-Formulation of Maxflow

$$\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
& f_{sy}\ (y \ne s,t): & 1\ell_{sy} - \ 1 + 1p_y & \ge & 0 \\
& f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x + \ 1 & \ge & 0 \\
& f_{ty}\ (y \ne s,t): & 1\ell_{ty} - \ 0 + 1p_y & \ge & 0 \\
& f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x + \ 0 & \ge & 0 \\
& f_{st}: & 1\ell_{st} - \ 1 + \ 0 & \ge & 0 \\
& f_{ts}: & 1\ell_{ts} - \ 0 + \ 1 & \ge & 0 \\
& & \ell_{xy} & \ge & 0
\end{array}$$

## LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \geq & 0 \\
& f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \geq & 0 \\
& f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
& f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.



## LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
& & p_s & = & 1 \\
& & p_t & = & 0
\end{array}
$$

We can interpret the $\ell_{xy}$ value as assigning a length to every edge.

The value $p_x$ for a variable, then can be seen as the distance of $x$ to $t$ (where the distance from $s$ to $t$ is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).



One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value $1$ on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.