Facility Location

Given a set L of (possible) locations for placing facilities and a set D of customers together with cost functions $s : D \times L \to \mathbb{R}^+$ and $o : L \to \mathbb{R}^+$ find a set of facility locations F together with an assignment $\phi : D \to F$ of customers to open facilities such that

$$\sum_{f \in F} o(f) + \sum_{c} s(c, \phi(c))$$

is minimized.

In the metric facility location problem we have

$$s(c, f) \le s(c, f') + s(c', f) + s(c', f') \ .$$

Facility Location

Integer Program

$$\begin{array}{llrcl}
\text{min} & & \multicolumn{3}{c}{\sum_{i\in F} f_i y_i + \sum_{i\in F}\sum_{j\in D} c_{ij} x_{ij}} \\
\text{s.t.} & \forall j \in D & \sum_{i\in F} x_{ij} & = & 1 \\
 & \forall i \in F, j \in D & x_{ij} & \le & y_i \\
 & \forall i \in F, j \in D & x_{ij} & \in & \{0,1\} \\
 & \forall i \in F & y_i & \in & \{0,1\}
\end{array}$$

As usual we get an LP by relaxing the integrality constraints.

Facility Location

Dual Linear Program

$$\begin{array}{llrcl}
\text{max} & & \multicolumn{3}{c}{\sum_{j\in D} v_j} \\
\text{s.t.} & \forall i \in F & \sum_{j\in D} w_{ij} & \le & f_i \\
 & \forall i \in F, j \in D & v_j - w_{ij} & \le & c_{ij} \\
 & \forall i \in F, j \in D & w_{ij} & \ge & 0
\end{array}$$

Facility Location

Definition 2

Given an LP solution (x^*, y^*) we say that facility i neighbours client j if $x_{ij} > 0$. Let $N(j) = \{i \in F : x^*_{ij} > 0\}$.

Lemma 3
If (x^*, y^*) is an optimal solution to the facility location LP and (v^*, w^*) is an optimal dual solution, then $x^*_{ij} > 0$ implies $c_{ij} \le v^*_j$.

Follows from slackness conditions.

Suppose we open set $S \subseteq F$ of facilities s.t. for all clients we have $S \cap N(j) \neq \emptyset$.

Then every client j has a facility i s.t. assignment cost for this client is at most $c_{ij} \le v^*_j$.

Hence, the total assignment cost is

$$\sum_j c_{i_j j} \le \sum_j v^*_j \le \mathrm{OPT} \ ,$$

where i_j is the facility that client j is assigned to.

Problem: Facility cost may be huge!

Suppose we can partition a subset $F' \subseteq F$ of facilities into neighbour sets of some clients. I.e.

$$F' = \biguplus_k N(j_k)$$

where $j_1, j_2, \ldots$ form a subset of the clients.

Now in each set $N(j_k)$ we open the cheapest facility. Call it f_{i_k}.

We have

$$f_{i_k} = f_{i_k} \sum_{i \in N(j_k)} x^*_{ij_k} \le \sum_{i \in N(j_k)} f_i x^*_{ij_k} \le \sum_{i \in N(j_k)} f_i y^*_i \ .$$

Summing over all k gives

$$\sum_k f_{i_k} \le \sum_k \sum_{i \in N(j_k)} f_i y^*_i = \sum_{i \in F'} f_i y^*_i \le \sum_{i \in F} f_i y^*_i$$

Facility cost is at most the facility cost in an optimum solution.

Problem: so far clients $j_1, j_2, \ldots$ have a neighboring facility. What about the others?

Definition 4

Let $N^2(j)$ denote all neighboring clients of the neighboring facilities of client j.

Note that $N(j)$ is a set of facilities while $N^2(j)$ is a set of clients.

Algorithm 1 FacilityLocation

$C \leftarrow D$// unassigned clients
$k \leftarrow 0$
while $C \neq 0$ **do**
    $k \leftarrow k + 1$
    choose $j_k \in C$ that minimizes v_j^*
    choose $i_k \in N(j_k)$ as cheapest facility
    assign j_k and all unassigned clients in $N^2(j_k)$ to i_k
    $C \leftarrow C - \{j_k\} - N^2(j_k)$

Facility cost of this algorithm is at most OPT because the sets $N(j_k)$ are disjoint.

Total assignment cost:

- Fix k; set $j = j_k$ and $i = i_k$. We know that $c_{ij} \leq v_j^*$.
- Let $\ell \in N^2(j)$ and h (one of) its neighbour(s) in $N(j)$.

$$c_{i\ell} \leq c_{ij} + c_{hj} + c_{h\ell} \leq v_j^* + v_j^* + v_\ell^* \leq 3v_\ell^*$$

Summing this over all facilities gives that the total assignment cost is at most $3 \cdot \text{OPT}$. Hence, we get a 4-approximation.

In the above analysis we use the inequality

$$\sum_{i \in F} f_i y_i^* \leq \text{OPT} \ .$$

We know something stronger namely

$$\sum_{i \in F} f_i y_i^* + \sum_{i \in F} \sum_{j \in D} c_{ij} x_{ij}^* \leq \text{OPT} \ .$$

Observation:

- Suppose when choosing a client j_k, instead of opening the cheapest facility in its neighborhood we choose a random facility according to $x^*_{ij_k}$.
- Then we incur connection cost

$$\sum_i c_{ij_k} x^*_{ij_k}$$

  for client j_k. (In the previous algorithm we estimated this by $v^*_{j_k}$).
- Define

$$C^*_j = \sum_i c_{ij} x^*_{ij}$$

  to be the connection cost for client j.

What will our facility cost be?

We only try to open a facility once (when it is in neighborhood of some j_k). (recall that neighborhoods of different $j_k' s$ are disjoint).

We open facility i with probability $x_{ij_k} \leq y_i$ (in case it is in some neighborhood; otw. we open it with probability zero).

Hence, the expected facility cost is at most

$$\sum_{i \in F} f_i y_i \enspace .$$

Algorithm 1 FacilityLocation

1: $C \leftarrow D$// unassigned clients
2: $k \leftarrow 0$
3: **while** $C \neq 0$ **do**
4: $\quad k \leftarrow k + 1$
5: $\quad$ choose $j_k \in C$ that minimizes $v^*_j + C^*_j$
6: $\quad$ choose $i_k \in N(j_k)$ according to probability x_{ij_k}.
7: $\quad$ assign j_k and all unassigned clients in $N^2(j_k)$ to i_k
8: $\quad C \leftarrow C - \{j_k\} - N^2(j_k)$

Total assignment cost:

- Fix k; set $j = j_k$.
- Let $\ell \in N^2(j)$ and h (one of) its neighbour(s) in $N(j)$.
- If we assign a client ℓ to the same facility as i we pay at most

$$\sum_i c_{ij} x^*_{ij_k} + c_{hj} + c_{h\ell} \leq C^*_j + v^*_j + v^*_\ell \leq C^*_\ell + 2v^*_\ell$$

Summing this over all clients gives that the total assignment cost is at most

$$\sum_j C^*_j + \sum_j 2v^*_j \leq \sum_j C^*_j + 2\mathrm{OPT}$$

Hence, it is at most $2\mathrm{OPT}$ plus the total assignment cost in an optimum solution.

Adding the facility cost gives a 3-approximation.