# 8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^T x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have $m$ constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If $d$ is much smaller than $m$ one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in $m$.

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**Setting:**

- We assume an LP of the form

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

- We assume that the LP is bounded.

# Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.**

# Computing a Lower Bound

Let $s$ denote the smallest common multiple of all denominators of entries in $A, b$.

Multiply entries in $A, b$ by $s$ to obtain integral entries. This does not change the feasible region.

Add slack variables to $A$; denote the resulting matrix with $\bar{A}$.

If $B$ is an optimal basis then $x_B$ with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are $0$).

**Theorem 2 (Cramers Rule)**

*Let $M$ be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by*

$$x_j = \frac{\det(M_j)}{\det(M)} \ ,$$

*where $M_j$ is the matrix obtained from $M$ by replacing the $j$-th column by the vector $b$.*

**Proof:**

- Define

$$X_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & x & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

  Note that expanding along the $j$-th column gives that $\det(X_j) = x_j$.

- Further, we have

$$MX_j = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{j-1} & Mx & Me_{j+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_j$$

- Hence,

$$x_j = \det(X_j) = \frac{\det(M_j)}{\det(M)}$$

# Bounding the Determinant

Let $Z$ be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or $c$. Let $C$ denote the matrix obtained from $\bar{A}_B$ by replacing the $j$-th column with vector $\bar{b}$.

Observe that

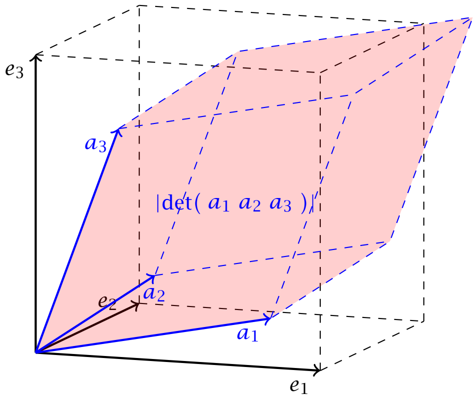
$$
\begin{aligned}
|\det(C)| &= \left| \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \prod_{1 \le i \le m} C_{i\pi(i)} \right| \\
&\le \sum_{\pi \in S_m} \prod_{1 \le i \le m} |C_{i\pi(i)}| \\
&\le m! \cdot Z^m \enspace .
\end{aligned}
$$

# Bounding the Determinant

Alternatively, Hadamards inequality gives

$$
\begin{aligned}
|\det(C)| &\le \prod_{i=1}^{m} \|C_{*i}\| \le \prod_{i=1}^{m} (\sqrt{m}Z) \\
&\le m^{m/2} Z^m \enspace .
\end{aligned}
$$

# Hadamards Inequality




Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

# Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.** Add the constraint $c^T x \geq -mZ(m! \cdot Z^m) - 1$. Note that this constraint is superfluous unless the LP is unbounded.

# Ensuring Conditions

Compute an optimum basis for the new LP.

- If the cost is $c^T x = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^T x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP$(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over $d$ variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint
$c^T x \geq -(mZ)(m! \cdot Z^m) - 1$.

**Algorithm 1** SeidelLP($\mathcal{H}, d$)

**if** $d = 1$ **then** solve 1-dimensional problem and return;
**if** $\mathcal{H} = \emptyset$ **then** return $x$ on implicit constraint hyperplane
choose random constraint $h \in \mathcal{H}$
$\hat{\mathcal{H}} \leftarrow \mathcal{H} \setminus \{h\}$
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d$)
**if** $\hat{x}^* =$ infeasible **then return** infeasible
**if** $\hat{x}^*$ fulfills $h$ **then return** $\hat{x}^*$
// optimal solution fulfills $h$ with equality, i.e., $a_h^T x = b_h$
solve $a_h^T x = b_h$ for some variable $x_\ell$;
eliminate $x_\ell$ in constraints from $\hat{\mathcal{H}}$ and in implicit constr.;
$\hat{x}^* \leftarrow$ SeidelLP($\hat{\mathcal{H}}, d - 1$)
**if** $\hat{x}^* =$ infeasible **then**
    **return** infeasible
**else**
    add the value of $x_\ell$ to $\hat{x}^*$ and return the solution

# 8 Seidels LP-algorithm

- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(m)$.
- If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return $d$-dimensional vector $x$.
- The first recursive call takes time $T(m-1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills $h$.
- If we are unlucky and $\hat{x}^*$ does not fulfill $h$ we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate $x_\ell$. Then we make a recursive call that takes time $T(m-1, d-1)$.
- The probability of being unlucky is at most $d/m$ as there are at most $d$ constraints whose removal will decrease the objective function

# 8 Seidels LP-algorithm

This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(m) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

# 8 Seidels LP-algorithm

Let $C$ be the largest constant in the $\mathcal{O}$-notations.

$$T(m,d) = \begin{cases} Cm & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m-1,d) + & \\ \frac{d}{m}(Cdm + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

# 8 Seidels LP-algorithm

Let $C$ be the largest constant in the $\mathcal{O}$-notations.

We show $T(m,d) \leq Cf(d)\max\{1,m\}$.

**$d = 1$:**

$$T(m,1) \leq Cm \leq Cf(1)\max\{1,m\} \text{ for } f(1) \geq 1$$

**$d > 1; m = 0$ :**

$$T(0,d) \leq \mathcal{O}(d) \leq Cd \leq Cf(d)\max\{1,m\} \text{ for } f(d) \geq d$$

**$d > 1; m = 1$ :**

$$\begin{aligned} T(1,d) &= \mathcal{O}(d) + T(0,d) + d\Big(\mathcal{O}(d) + T(0,d-1)\Big) \\ &\leq Cd + Cd + Cd^2 + dCf(d-1) \\ &\leq Cf(d)\max\{1,m\} \text{ for } f(d) \geq 3d^2 + df(d-1) \end{aligned}$$

# 8 Seidels LP-algorithm

$\boldsymbol{d > 1; m > 1:}$

(by induction hypothesis statm. true for $d' < d, m' \geq 0$;
and for $d' = d$, $m' < m$)

$$\begin{aligned}
T(m,d) &= \mathcal{O}(d) + T(m-1,d) + \frac{d}{m}\Big(\mathcal{O}(dm) + T(m-1,d-1)\Big) \\
&\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m}Cf(d-1)(m-1) \\
&\leq 2Cd^2 + Cf(d)(m-1) + dCf(d-1) \\
&\leq Cf(d)m
\end{aligned}$$

if $f(d) \geq df(d-1) + 2d^2$.

# 8 Seidels LP-algorithm

- Define $f(1) = 3 \cdot 1^2$ and $f(d) = df(d-1) + 3d^2$ for $d > 1$.

Then

$$\begin{aligned}
f(d) &= 3d^2 + df(d-1) \\
&= 3d^2 + d\Big[3(d-1)^2 + (d-1)f(d-2)\Big] \\
&= 3d^2 + d\Big[3(d-1)^2 + (d-1)\Big[3(d-2)^2 + (d-2)f(d-3)\Big]\Big] \\
&= 3d^2 + 3d(d-1)^2 + 3d(d-1)(d-2)^2 + \dots \\
&\quad + 3d(d-1)(d-2) \cdot \ldots \cdot 4 \cdot 3 \cdot 2 \cdot 1^2 \\
&= 3d!\left(\frac{d^2}{d!} + \frac{(d-1)^2}{(d-1)!} + \frac{(d-2)^2}{(d-2)!} + \dots\right) \\
&= \mathcal{O}(d!)
\end{aligned}$$

since $\sum_{i\ge 1} \frac{i^2}{i!}$ is a constant.