

LP Feasibility Problem (LP feasibility)

- ▶ Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?
- ▶ Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the **feasible region** does not change.

Is this problem in NP or even in P?

The Bit Model

Input size

- ▶ The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- ▶ Let for an $m \times n$ matrix M , $L(M)$ denote the number of bits required to encode all the numbers in M .

$$L(M) := \sum_{i,j} \lceil \log_2(|m_{ij}|) \rceil + 1$$

- ▶ In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- ▶ Then the input length is $\Theta(L([A|b]))$.

- ▶ In the following we sometimes refer to $L := L([A|b])$ as the input size (even though the real input size is something in $\Theta(L([A|b]))$).
- ▶ In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in $L([A|b])$).

Suppose that $Ax = b$; $x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = A_B^{-1}b$$

and all other entries in x are 0.

Size of a Basic Feasible Solution

Lemma 2

Let $M \in \mathbb{Z}^{m \times m}$ be an invertible matrix and let $b \in \mathbb{Z}^m$. Further define $L' = L([M \mid b]) + n \log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \leq 2^{L'}$ and $|D| \leq 2^{L'}$.

Proof:

Cramer's rule says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

where M_j is the matrix obtained from M by replacing the j -th column by the vector b .

Bounding the Determinant

Let $X = A_B$. Then

$$\begin{aligned} |\det(X)| &= \left| \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{1 \leq i \leq n} X_{i\pi(i)} \right| \\ &\leq \sum_{\pi \in S_n} \prod_{1 \leq i \leq n} |X_{i\pi(i)}| \\ &\leq n! \cdot 2^{L([A|b])} \leq n^n 2^L \leq 2^{L'} . \end{aligned}$$

Analogously for $\det(M_j)$.

This means if $Ax = b$, $x \geq 0$ is feasible we only need to consider vectors x where an entry x_j can be represented by a rational number with encoding length polynomial in the input length L .

Hence, the x that we have to guess is of length polynomial in the input-length L .

For a given vector x of polynomial length we can check for feasibility in polynomial time.

Hence, LP feasibility is in NP.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^T x \mid Ax = b; x \geq 0\}$ do a **binary search** for the optimum solution

(Add constraint $c^T x - \delta = M; \delta \geq 0$ or $(c^T x \geq M)$. Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2 \left(\frac{2n2^{2L'}}{1/2^{L'}} \right) = \mathcal{O}(L') ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

Here we use $L' = L([A \mid b \mid c]) + n \log_2 n$ (it also includes the encoding size of c).

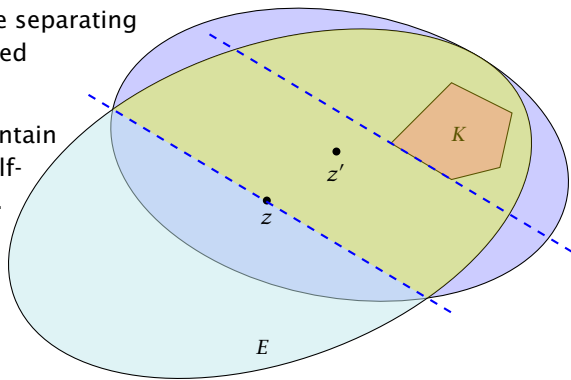
How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a **basic feasible solution**.

We can add a constraint $c^T x \geq M_{\max} + 1$ and check for feasibility.

Ellipsoid Method

- ▶ Let K be a convex set.
- ▶ Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
- ▶ If center $z \in K$ STOP.
- ▶ Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).
- ▶ Shift hyperplane to contain node z . H denotes half-space that contains K .
- ▶ Compute (smallest) ellipsoid E' that contains $K \cap H$.
- ▶ REPEAT



Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ What if the polytop K is unbounded?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

Definition 3

A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an **affine transformation**.

Definition 4

A ball in \mathbb{R}^n with center c and radius r is given by

$$\begin{aligned} B(c, r) &= \{x \mid (x - c)^T (x - c) \leq r^2\} \\ &= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\} \end{aligned}$$

$B(0, 1)$ is called the **unit ball**.

Definition 5

An affine transformation of the unit ball is called an **ellipsoid**.

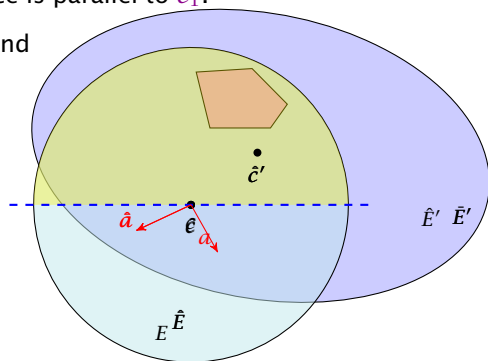
From $f(x) = Lx + t$ follows $x = L^{-1}(f(x) - t)$.

$$\begin{aligned} f(B(0, 1)) &= \{f(x) \mid x \in B(0, 1)\} \\ &= \{y \in \mathbb{R}^n \mid L^{-1}(y - t) \in B(0, 1)\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^T L^{-1T} L^{-1}(y - t) \leq 1\} \\ &= \{y \in \mathbb{R}^n \mid (y - t)^T Q^{-1}(y - t) \leq 1\} \end{aligned}$$

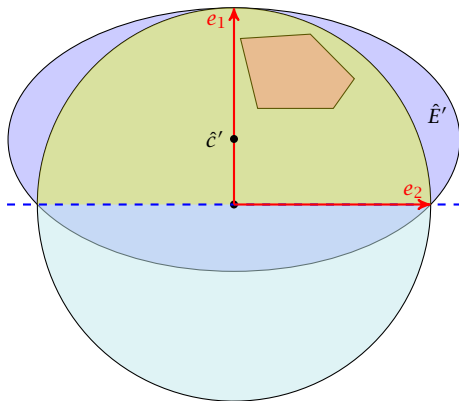
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid

- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- ▶ Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1 .
- ▶ Compute the new center \hat{c}' and the new matrix \hat{Q}' for this simplified setting.
- ▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E .



The Easy Case



- ▶ The new center lies on axis x_1 . Hence, $\hat{c}' = te_1$ for $t > 0$.
- ▶ The vectors e_1, e_2, \dots have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$.

The Easy Case

- ▶ To obtain the matrix \hat{Q}'^{-1} for our ellipsoid \hat{E}' note that \hat{E}' is **axis-parallel**.
- ▶ Let a denote the radius along the x_1 -axis and let b denote the (common) radius for the other axes.
- ▶ The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

The Easy Case

- ▶ As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix \hat{Q}'^{-1} is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- ▶ $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $(1-t)^2 = a^2$.

The Easy Case

- ▶ For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ gives

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

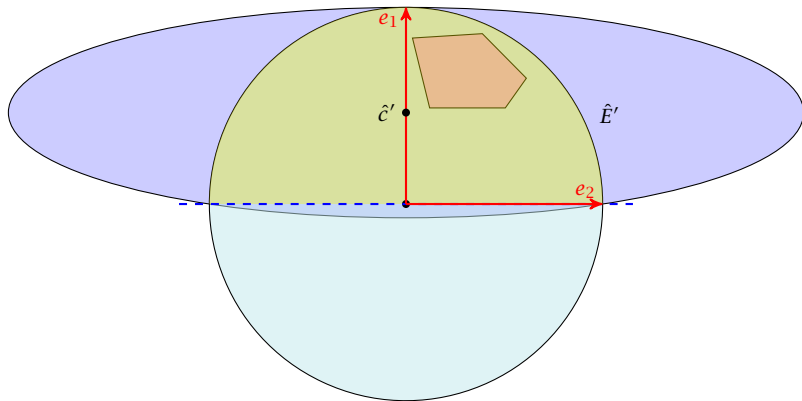
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

The Easy Case

We still have many choices for t :



Choose t such that the volume of \hat{E}' is minimal!!!

The Easy Case

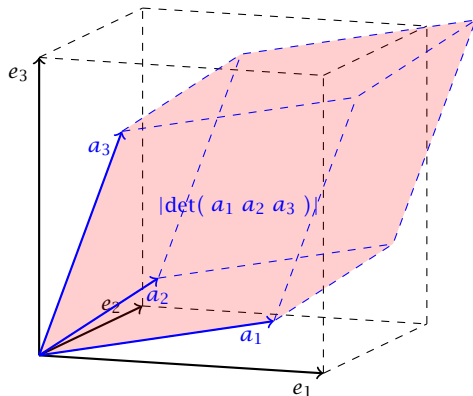
We want to choose t such that the volume of \hat{E}' is minimal.

Lemma 6

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\text{vol}(L(K)) = |\det(L)| \cdot \text{vol}(K) .$$

n-dimensional volume



The Easy Case

- ▶ We want to choose t such that the volume of \hat{E}' is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0,1)) \cdot |\det(\hat{L}')| ,$$

where $\hat{Q}' = \hat{L}'\hat{L}'^T$.

- ▶ We have

$$\hat{L}'^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & \dots & 0 \\ 0 & \frac{1}{b} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b} \end{pmatrix} \text{ and } \hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- ▶ Note that a and b in the above equations depend on t , by the previous equations.

The Easy Case

$$\begin{aligned}\text{vol}(\hat{E}') &= \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| \\ &= \text{vol}(B(0, 1)) \cdot ab^{n-1} \\ &= \text{vol}(B(0, 1)) \cdot (1-t) \cdot \left(\frac{1-t}{\sqrt{1-2t}}\right)^{n-1} \\ &= \text{vol}(B(0, 1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\end{aligned}$$

The Easy Case

$$\begin{aligned}
 \frac{d \operatorname{vol}(\hat{E}')}{dt} &= \frac{d}{dt} \left(\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\
 &= \frac{1}{N^2} \cdot \left(\overbrace{(-1) \cdot n(1-t)^{n-1}}^{\text{derivative of numerator}} \cdot \overbrace{(\sqrt{1-2t})^{n-1}}^{\text{denominator}} \right. \\
 &\quad \left. \cdot \overbrace{(n-1)(\sqrt{1-2t})^{n-2}}^{\text{outer derivative}} \cdot \overbrace{\frac{1}{2\sqrt{1-2t}} \cdot (-2)}^{\text{inner derivative}} \cdot \overbrace{(1-t)^n}^{\text{numerator}} \right) \\
 &= \frac{1}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left((n-1)(1-t) - n(1-2t) \right) \\
 &= \frac{1}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left((n+1)t - 1 \right)
 \end{aligned}$$

The Easy Case

- ▶ We obtain the minimum for $t = \frac{1}{n+1}$.
- ▶ For this value we obtain

$$a = 1 - t = \frac{n}{n+1} \text{ and } b = \frac{1-t}{\sqrt{1-2t}} = \frac{n}{\sqrt{n^2-1}}$$

To see the equation for b , observe that

$$b^2 = \frac{(1-t)^2}{1-2t} = \frac{\left(1 - \frac{1}{n+1}\right)^2}{1 - \frac{2}{n+1}} = \frac{\left(\frac{n}{n+1}\right)^2}{\frac{n-1}{n+1}} = \frac{n^2}{n^2-1}$$

The Easy Case

Let $\gamma_n = \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

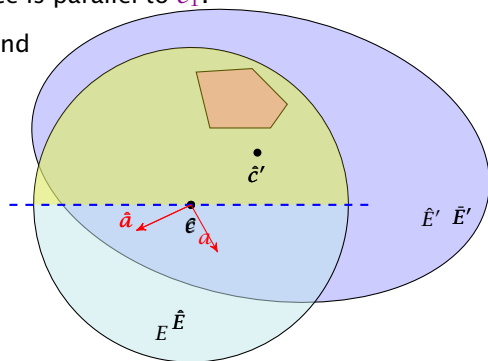
$$\begin{aligned}\gamma_n^2 &= \left(\frac{n}{n+1}\right)^2 \left(\frac{n^2}{n^2-1}\right)^{n-1} \\ &= \left(1 - \frac{1}{n+1}\right)^2 \left(1 + \frac{1}{(n-1)(n+1)}\right)^{n-1} \\ &\leq e^{-2\frac{1}{n+1}} \cdot e^{\frac{1}{n+1}} \\ &= e^{-\frac{1}{n+1}}\end{aligned}$$

where we used $(1+x)^a \leq e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

This gives $\gamma_n \leq e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid

- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- ▶ Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1 .
- ▶ Compute the new center \hat{c}' and the new matrix \hat{Q}' for this simplified setting.
- ▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E .



Our progress is the same:

$$\begin{aligned} e^{-\frac{1}{2(n+1)}} &\geq \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = \frac{\text{vol}(\hat{E}')}{\text{vol}(\hat{E})} = \frac{\text{vol}(R(\hat{E}'))}{\text{vol}(R(\hat{E}))} \\ &= \frac{\text{vol}(\bar{E}')}{\text{vol}(\bar{E})} = \frac{\text{vol}(f(\bar{E}'))}{\text{vol}(f(\bar{E}))} = \frac{\text{vol}(E')}{\text{vol}(E)} \end{aligned}$$

Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

How to Compute The New Parameters?

The transformation function of the (old) ellipsoid: $f(x) = Lx + c$;

The halfspace to be intersected: $H = \{x \mid a^T(x - c) \leq 0\}$;

$$\begin{aligned}f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \leq 0\} \\&= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \leq 0\} \\&= \{y \mid a^T(f(y) - c) \leq 0\} \\&= \{y \mid a^T(Ly + c - c) \leq 0\} \\&= \{y \mid (a^T L)y \leq 0\}\end{aligned}$$

This means $\bar{a} = L^T a$.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1 -direction. Hence,

$$R^{-1}\left(\frac{L^T a}{\|L^T a\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

$$\tilde{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1} \frac{L^T a}{\|L^T a\|}$$

$$\begin{aligned}c' &= f(\tilde{c}') = L \cdot \tilde{c}' + c \\&= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\&= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}\end{aligned}$$

For computing the matrix Q' of the new ellipsoid we assume in the following that \hat{E}' , \bar{E}' and E' refer to the ellipsoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left(I - \frac{2}{n + 1} e_1 e_1^T \right)$$

because for $a = n/n+1$ and $b = n/\sqrt{n^2-1}$

$$\begin{aligned} b^2 - b^2 \frac{2}{n + 1} &= \frac{n^2}{n^2 - 1} - \frac{2n^2}{(n - 1)(n + 1)^2} \\ &= \frac{n^2(n + 1) - 2n^2}{(n - 1)(n + 1)^2} = \frac{n^2(n - 1)}{(n - 1)(n + 1)^2} = a^2 \end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \leq 1\} \\ &= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1}y \leq 1\} \\ &= \{y \mid y^T (R^T)^{-1} \hat{Q}'^{-1} R^{-1}y \leq 1\} \\ &= \{y \mid y^T \underbrace{(R\hat{Q}'R^T)^{-1}}_{\hat{Q}'} y \leq 1\}\end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1} \left(I - \frac{2}{n+1} e_1 e_1^T \right) \cdot R^T \\ &= \frac{n^2}{n^2-1} \left(R \cdot R^T - \frac{2}{n+1} (Re_1)(Re_1)^T \right) \\ &= \frac{n^2}{n^2-1} \left(I - \frac{2}{n+1} \frac{L^T a a^T L}{\|L^T a\|^2} \right)\end{aligned}$$

9 The Ellipsoid Algorithm

$$\begin{aligned} E' &= L(\bar{E}') \\ &= \{L(x) \mid x^T \bar{Q}'^{-1} x \leq 1\} \\ &= \{y \mid (L^{-1}y)^T \bar{Q}'^{-1} L^{-1}y \leq 1\} \\ &= \{y \mid y^T (L^T)^{-1} \bar{Q}'^{-1} L^{-1}y \leq 1\} \\ &= \{y \mid y^T \underbrace{(L\bar{Q}'L^T)^{-1}}_{Q'} y \leq 1\} \end{aligned}$$

9 The Ellipsoid Algorithm

Hence,

$$\begin{aligned} Q' &= L\bar{Q}'L^T \\ &= L \cdot \frac{n^2}{n^2-1} \left(I - \frac{2}{n+1} \frac{L^T a a^T L}{a^T Q a} \right) \cdot L^T \\ &= \frac{n^2}{n^2-1} \left(Q - \frac{2}{n+1} \frac{Q a a^T Q}{a^T Q a} \right) \end{aligned}$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
- 2: **output:** point $x \in K$ or “ K is empty”
- 3: $Q \leftarrow ???$
- 4: **repeat**
- 5: **if** $c \in K$ **then return** c
- 6: **else**
- 7: choose a violated hyperplane a
- 8:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 9:
$$Q \leftarrow \frac{n^2}{n^2-1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 10: **endif**
- 11: **until** $???$
- 12: **return** “ K is empty”

Repeat: Size of basic solutions

Lemma 7

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $\langle a_{\max} \rangle$ be the maximum encoding length of an entry in A, b . Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^{2n\langle a_{\max} \rangle + 2n \log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max} \rangle + 2n \log_2 n}$.

Note that here we have $P = \{x \mid Ax \leq b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \\ -A & A & \end{bmatrix}$, $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices \bar{A}_B and \bar{M}_j (matrix obt. when replacing the j -th column of \bar{A}_B by \bar{b}) can become at most

$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^{2n} \\ &\leq (\sqrt{2n} \cdot 2^{\langle a_{\max} \rangle})^{2n} \leq 2^{2n\langle a_{\max} \rangle + 2n \log_2 n}, \end{aligned}$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from \bar{A} .

This holds because columns from I_m selected when going from \bar{A} to \bar{A}_B do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that \bar{A} consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n B(0, 1) \leq (n\delta)^n B(0, 1)$.

When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b .

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where $\lambda = \delta^2 + 1$.

Lemma 8

P_λ is feasible if and only if P is feasible.

\Leftarrow : obvious!

\Rightarrow :

Consider the polyhedrons

$$\bar{P} = \left\{ x \mid \begin{bmatrix} A & -A & I_m \\ -A & A & 0 \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}; x \geq 0 \right\}$$

and

$$\bar{P}_\lambda = \left\{ x \mid \begin{bmatrix} A & -A & I_m \\ -A & A & 0 \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0 \right\} .$$

P is feasible if and only if \bar{P} is feasible, and P_λ feasible if and only if \bar{P}_λ feasible.

\bar{P}_λ is bounded since P_λ and P are bounded.

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$, and $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$.

\bar{P}_λ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1} \bar{b} + \frac{1}{\lambda} \bar{A}_B^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x -values are zero)

The only reason that this basic feasible solution is not feasible for \bar{P} is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1} \bar{b})_i < 0 \leq (\bar{A}_B^{-1} \bar{b})_i + \frac{1}{\lambda} (\bar{A}_B^{-1} \vec{1})_i$$

By Cramers rule we get

$$(\bar{A}_B^{-1} \bar{b})_i < 0 \quad \Rightarrow \quad (\bar{A}_B^{-1} \bar{b})_i \leq -\frac{1}{\det(\bar{A}_B)}$$

and

$$(\bar{A}_B^{-1} \vec{1})_i \leq \det(\bar{M}_j) ,$$

where \bar{M}_j is obtained by replacing the j -th column of \bar{A}_B by $\vec{1}$.

However, we showed that the determinants of \bar{A}_B and \bar{M}_j can become at most δ .

Since, we chose $\lambda = \delta^2 + 1$ this gives a contradiction.

Lemma 9

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$.

Proof:

If P_λ feasible then also P . Let x be feasible for P .

This means $Ax \leq b$.

Let $\vec{\ell}$ with $\|\vec{\ell}\| \leq r$. Then

$$\begin{aligned}(A(x + \vec{\ell}))_i &= (Ax)_i + (A\vec{\ell})_i \leq b_i + \vec{a}_i^T \vec{\ell} \\ &\leq b_i + \|\vec{a}_i\| \cdot \|\vec{\ell}\| \leq b_i + \sqrt{n} \cdot 2^{\langle a_{\max} \rangle} \cdot r \\ &\leq b_i + \frac{\sqrt{n} \cdot 2^{\langle a_{\max} \rangle}}{\delta^3} \leq b_i + \frac{1}{\delta^2 + 1} \leq b_i + \frac{1}{\lambda}\end{aligned}$$

Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

How many iterations do we need until the volume becomes too small?

$$e^{-\frac{i}{2(n+1)}} \cdot \text{vol}(B(0, R)) < \text{vol}(B(0, r))$$

Hence,

$$\begin{aligned} i &> 2(n+1) \ln\left(\frac{\text{vol}(B(0, R))}{\text{vol}(B(0, r))}\right) \\ &= 2(n+1) \ln\left(n^n \delta^n \cdot \delta^{3n}\right) \\ &= 8n(n+1) \ln(\delta) + 2(n+1)n \ln(n) \\ &= \mathcal{O}(\text{poly}(n, \langle a_{\max} \rangle)) \end{aligned}$$

Algorithm 1 ellipsoid-algorithm

- 1: **input:** point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
- 2: with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
- 3: **output:** point $x \in K$ or “ K is empty”
- 4: $Q \leftarrow \text{diag}(R^2, \dots, R^2)$ // i.e., $L = \text{diag}(R, \dots, R)$
- 5: **repeat**
- 6: **if** $c \in K$ **then return** c
- 7: **else**
- 8: choose a violated hyperplane a
- 9:
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
- 10:
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
- 11: **endif**
- 12: **until** $\det(Q) \leq r^{2n}$ // i.e., $\det(L) \leq r^n$
- 13: **return** “ K is empty”

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- ▶ certifies that $x \in K$,
- ▶ or finds a hyperplane separating x from K .

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- ▶ a guarantee that a ball of radius r is contained in K ,
- ▶ an initial ball $B(c, R)$ with radius R that contains K ,
- ▶ a separation oracle for K .

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.