

Bin Packing

Given n items with sizes s_1, \dots, s_n where

$$1 > s_1 \geq \dots \geq s_n > 0 .$$

Pack items into a minimum number of bins where each bin can hold items of total size at most 1.

Theorem 5

There is no ρ -approximation for Bin Packing with $\rho < 3/2$ unless $P = NP$.

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Proof

- ▶ In the partition problem we are given positive integers b_1, \dots, b_n with $B = \sum_i b_i$ even. Can we partition the integers into two sets S and T s.t.

$$\sum_{i \in S} b_i = \sum_{i \in T} b_i \quad ?$$

- ▶ We can solve this problem by setting $s_i := 2b_i/B$ and asking whether we can pack the resulting items into 2 bins or not.
- ▶ A ρ -approximation algorithm with $\rho < 3/2$ cannot output 3 or more bins when 2 are optimal.
- ▶ Hence, such an algorithm can solve Partition.

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Definition 6

An asymptotic polynomial-time approximation scheme (APTAS) is a family of algorithms $\{A_\epsilon\}$ along with a constant c such that A_ϵ returns a solution of value at most $(1 + \epsilon)\text{OPT} + c$ for minimization problems.

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Bin Packing

Again we can differentiate between small and large items.

Lemma 7

Any packing of items into ℓ bins can be extended with items of size at most γ s.t. we use only $\max\{\ell, \frac{1}{1-\gamma}\text{SIZE}(I) + 1\}$ bins, where $\text{SIZE}(I) = \sum_i s_i$ is the sum of all item sizes.

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- ▶ If after Greedy we use more than ℓ bins, all bins (apart from the last) must be full to at least $1 - \gamma$.
- ▶ Hence, $r(1 - \gamma) \leq \text{SIZE}(I)$ where r is the number of nearly-full bins.
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Choose $\gamma = \epsilon/2$. Then we either use ℓ bins or at most

$$\frac{1}{1 - \epsilon/2} \cdot \text{OPT} + 1 \leq (1 + \epsilon) \cdot \text{OPT} + 1$$

bins.

It remains to find an algorithm for the large items.

Linear Grouping:

Generate an instance I' (for large items) as follows.

- ▶ Order large items according to size.
- ▶ Let the first k items belong to group 1; the following k items belong to group 2; etc.
- ▶ Delete items in the first group;
- ▶ Round items in the remaining groups to the size of the largest item in the group.

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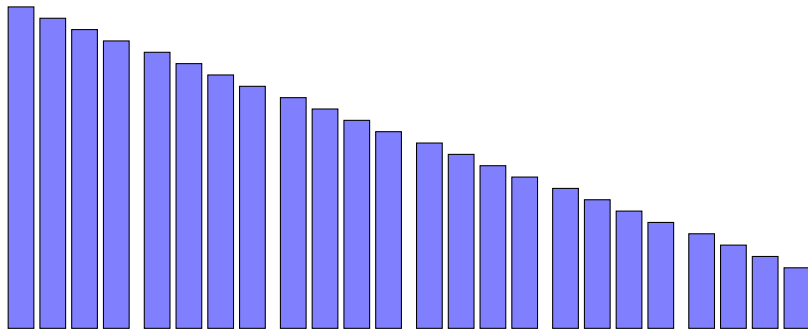
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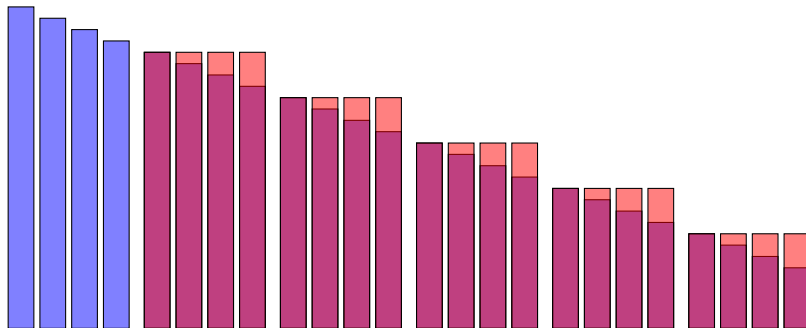
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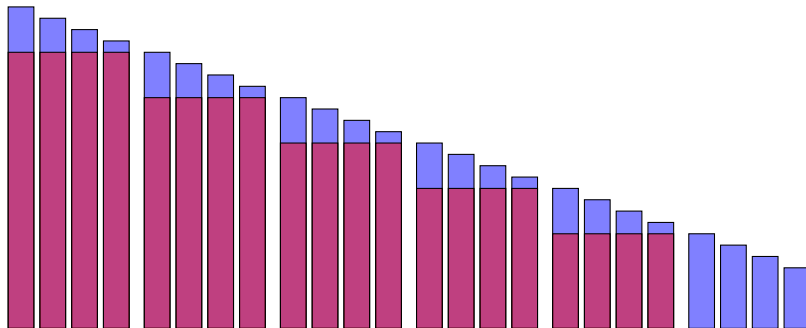
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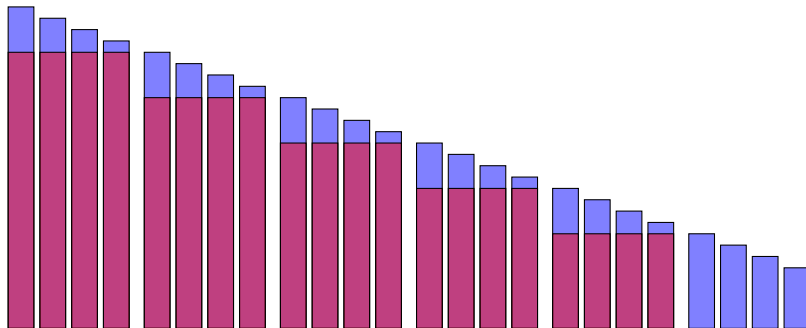
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Lemma 8

$$\text{OPT}(I') \leq \text{OPT}(I) \leq \text{OPT}(I') + k$$

Proof 1:

Any bin packing for I' gives a bin packing for I as follows:

pack the items of group 1 into the bins for I' (the bins for group 1 have been packed),

pack the items of groups 2, where in the packing for I' the bins for group 1 have been packed,

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Proof 2:

- ▶ Any bin packing for I' gives a bin packing for I as follows.
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Assume that our instance does not contain pieces smaller than $\epsilon/2$. Then $\text{SIZE}(I) \geq \epsilon n/2$.

We set $k = \lfloor \epsilon \text{SIZE}(I) \rfloor$.

Then $n/k \leq n/\lfloor \epsilon^2 n/2 \rfloor \leq 4/\epsilon^2$ (here we used $\lfloor \alpha \rfloor \geq \alpha/2$ for $\alpha \geq 1$).

Hence, after grouping we have a constant number of piece sizes ($4/\epsilon^2$) and at most a constant number ($2/\epsilon$) can fit into any bin.

We can find an optimal packing for such instances by the previous Dynamic Programming approach.

- ▶ cost (for large items) at most

$$\text{OPT}(I') + k \leq \text{OPT}(I) + \epsilon \text{SIZE}(I) \leq (1 + \epsilon) \text{OPT}(I)$$

- ▶ running time $\mathcal{O}((\frac{2}{\epsilon}n)^{4/\epsilon^2})$.

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