5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

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Dual:

$$\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \geq c, y \geq 0\}$$

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$$\min\{[b^T \; -b^T] y \mid [A^T \; -A^T] y \ge c, y \ge 0\}$$
$$= \min\left\{[b^T \; -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T \; -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \ge c, y^- \ge 0, y^+ \ge 0\right\}$$

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$$= \min\left\{b^T y' \,\middle|\, A^T y' \geq c\right\}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \geq c\}$.

Hence, the solution is optimal.

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\begin{aligned}
b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\
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