Duality

How do we get an upper bound to a maximization LP?

$$
\begin{array}{rrcrl}
\max & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \le 480 \\
 & 4a & + & 4b & \le 160 \\
 & 35a & + & 20b & \le 1190 \\
 & & & a, b & \ge 0
\end{array}
$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 2
Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the dual problem.

Duality

Lemma 3

The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$
- $w = -\max\{-b^T y \mid -A^T y \le -c, y \ge 0\}$

The dual problem is

- $z = -\min\{-c^T x \mid -Ax \ge -b, x \ge 0\}$
- $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

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Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

Theorem 4 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

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Weak Duality

$A^T\hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c$ $(\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b$ $(\hat{y} \geq 0)$

This gives

$$c^T\hat{x} \leq \hat{y}^T A\hat{x} \leq b^T\hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^T y = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$$A^T\hat{y} \geq c \Rightarrow \hat{x}^TA^T\hat{y} \geq \hat{x}^Tc \;(\hat{x} \geq 0)$$

$$A\hat{x} \leq b \Rightarrow y^TA\hat{x} \leq \hat{y}^Tb \;(\hat{y} \geq 0)$$

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If P is unbounded then D is infeasible.

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.