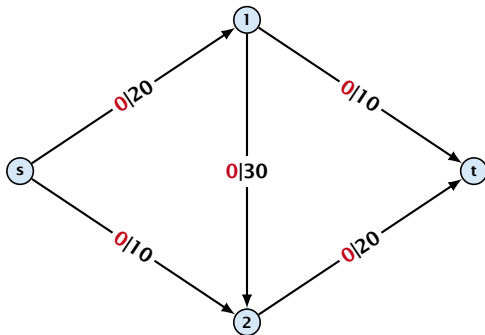


11 Augmenting Path Algorithms

Greedy-algorithm:

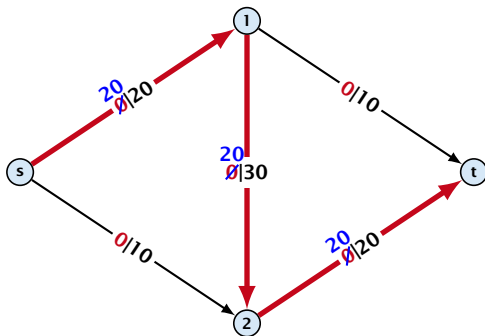
- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s - t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible



11 Augmenting Path Algorithms

Greedy-algorithm:

- ▶ start with $f(e) = 0$ everywhere
- ▶ find an s - t path with $f(e) < c(e)$ on every edge
- ▶ augment flow along the path
- ▶ repeat as long as possible



The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- ▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v .

The Residual Graph

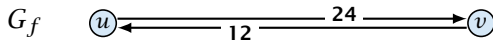
From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- ▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v .
- ▶ G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- ▶ Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v .
- ▶ G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.



Augmenting Path Algorithm

Definition 1

An **augmenting path** with respect to flow f , is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 1 FordFulkerson($G = (V, E, c)$)

- 1: Initialize $f(e) \leftarrow 0$ for all edges.
- 2: **while** \exists augmenting path p in G_f **do**
- 3: augment as much flow along p as possible.

Augmenting Path Algorithm

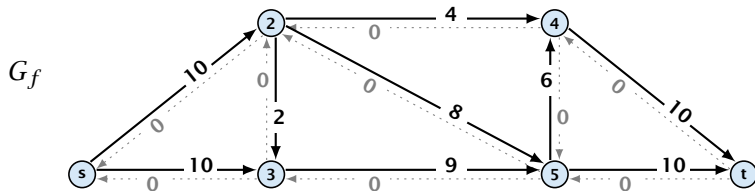
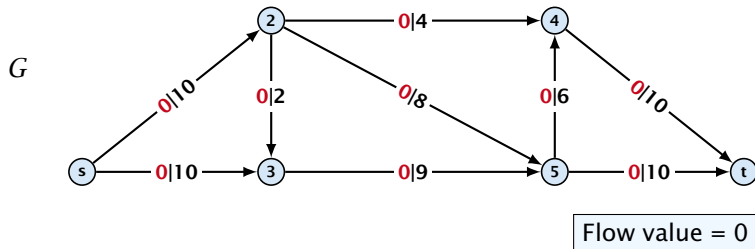
Definition 1

An **augmenting path** with respect to flow f , is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

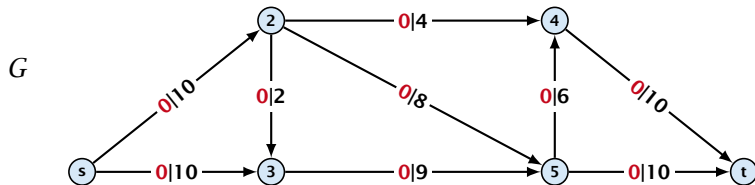
Algorithm 1 FordFulkerson($G = (V, E, c)$)

- 1: Initialize $f(e) \leftarrow 0$ for all edges.
- 2: **while** \exists augmenting path p in G_f **do**
- 3: augment as much flow along p as possible.

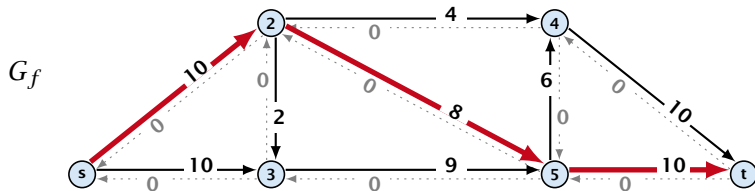
Augmenting Path Algorithm



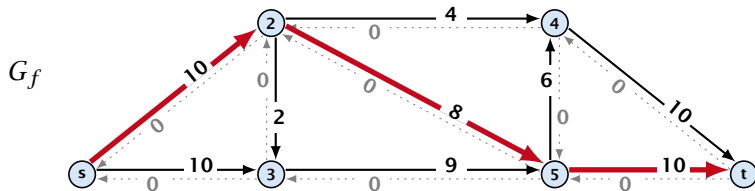
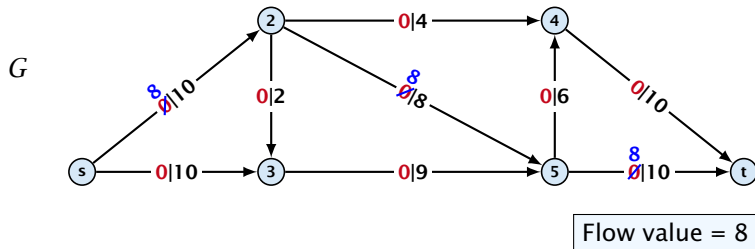
Augmenting Path Algorithm



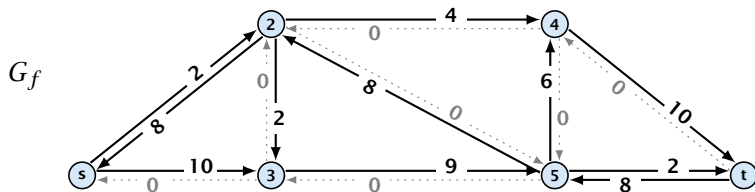
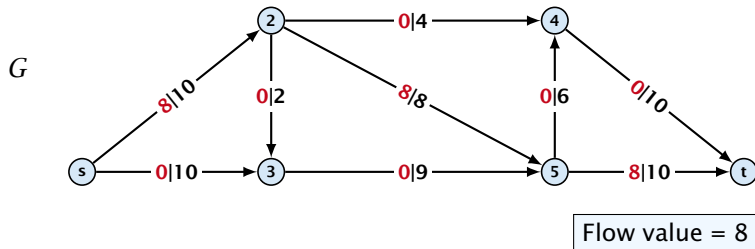
Flow value = 0



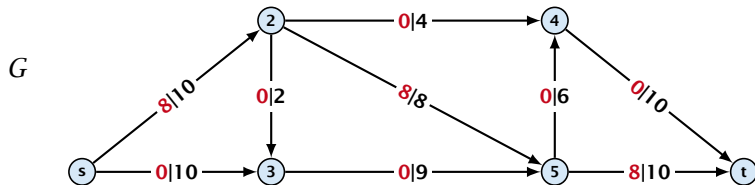
Augmenting Path Algorithm



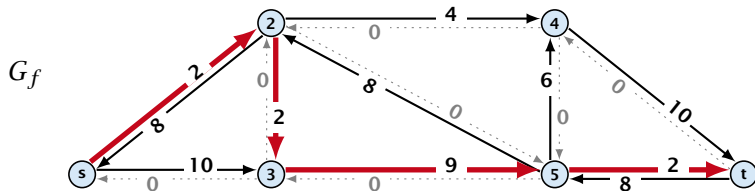
Augmenting Path Algorithm



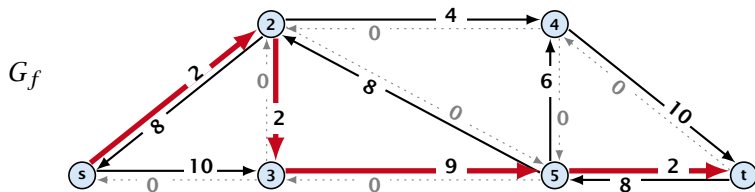
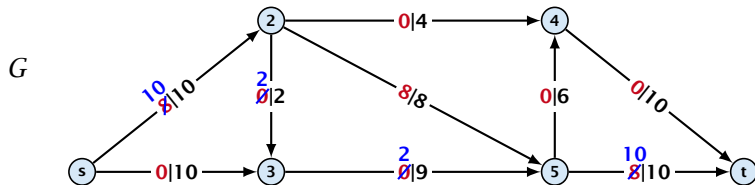
Augmenting Path Algorithm



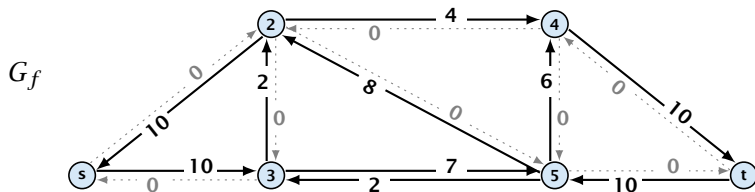
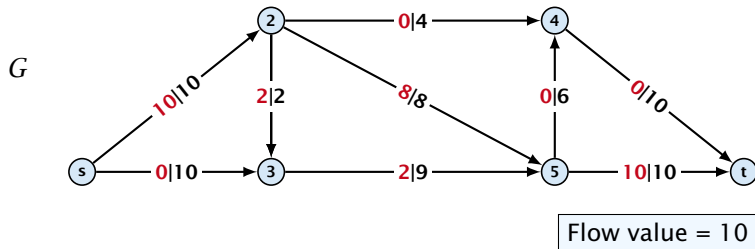
Flow value = 8



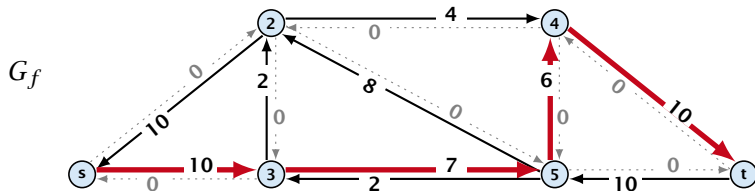
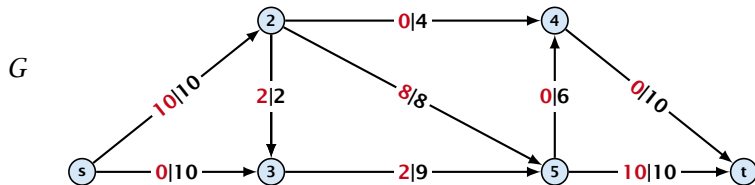
Augmenting Path Algorithm



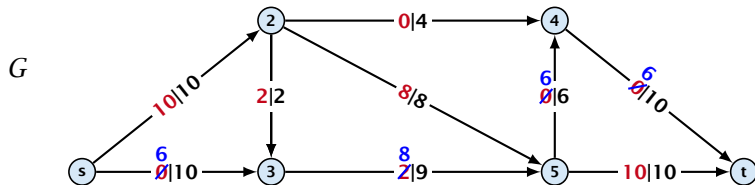
Augmenting Path Algorithm



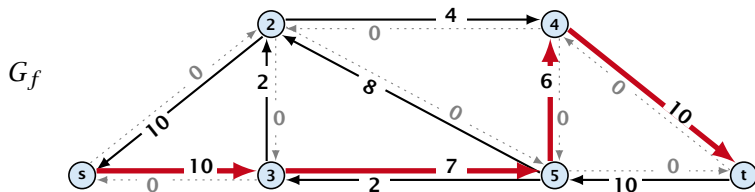
Augmenting Path Algorithm



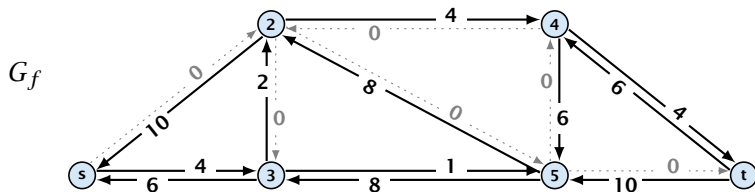
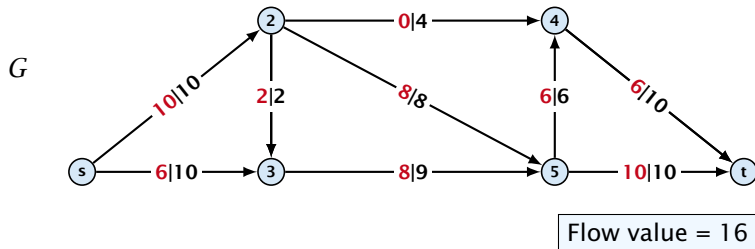
Augmenting Path Algorithm



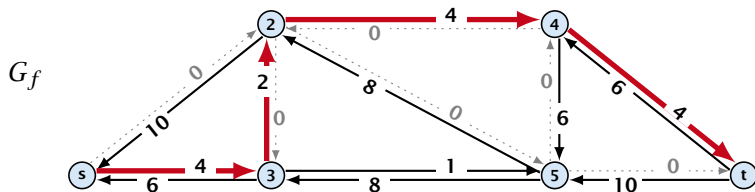
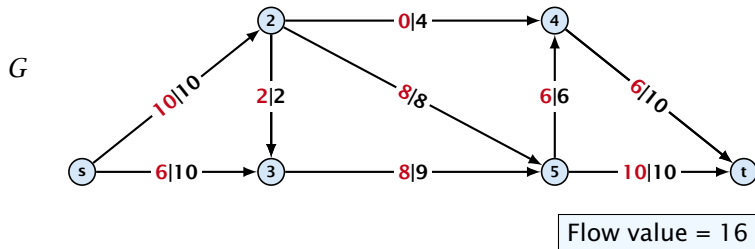
Flow value = 16



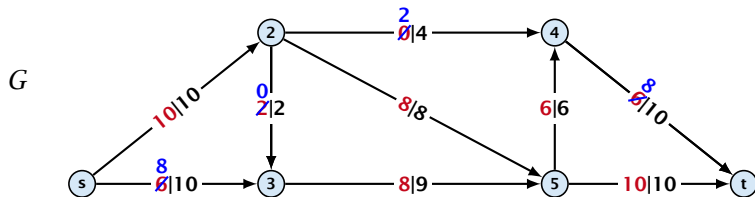
Augmenting Path Algorithm



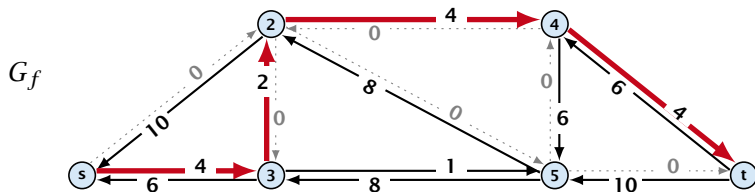
Augmenting Path Algorithm



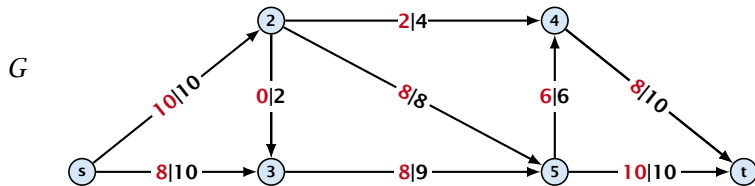
Augmenting Path Algorithm



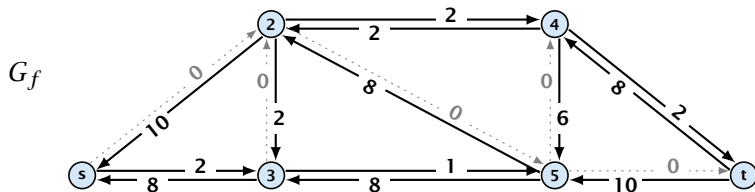
Flow value = 18



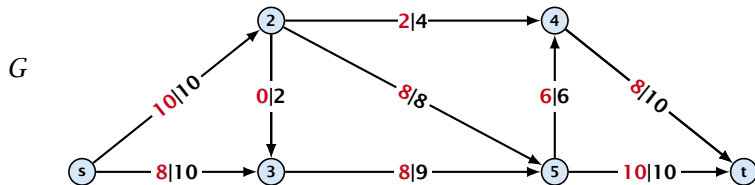
Augmenting Path Algorithm



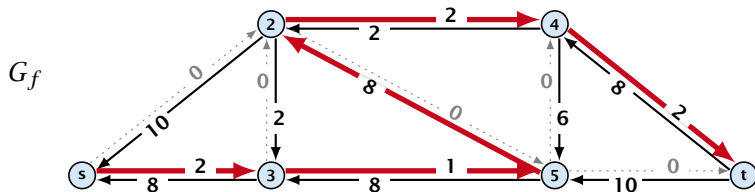
Flow value = 18



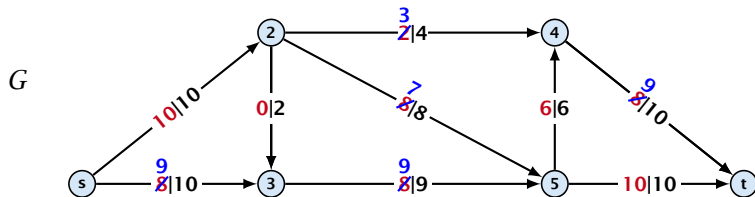
Augmenting Path Algorithm



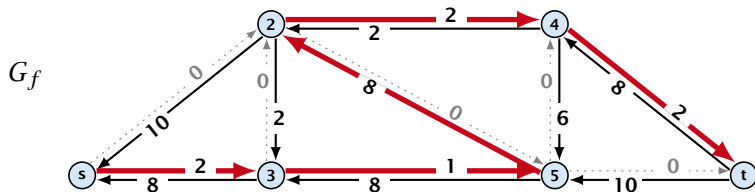
Flow value = 18



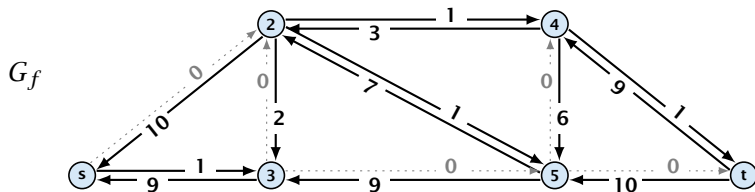
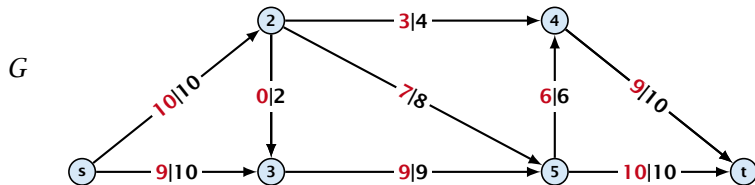
Augmenting Path Algorithm



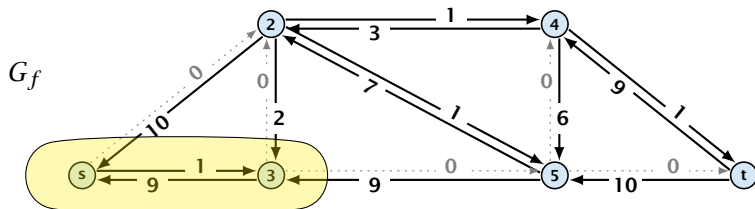
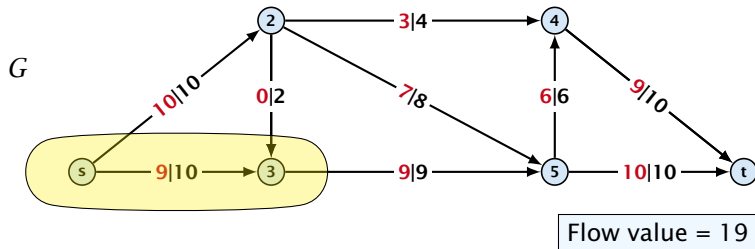
Flow value = 19



Augmenting Path Algorithm



Augmenting Path Algorithm



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow iff there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

- 1. There exists a cut (S, T) such that f is saturated w.r.t. (S, T) .
- 2. f is a maximum flow.
- 3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut (S, T) such that f is a maximum flow.
2. f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

- 1. There exists a cut (S, T) such that $|f| = c(S, T)$.
- 2. f is a maximum flow.
- 3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

Theorem 2

A flow f is a maximum flow **iff** there are no augmenting paths.

Theorem 3

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f .



Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

Let G_f be a flow with no augmenting paths.

Let S be the set of vertices reachable from s in the residual graph along non-zero capacity edges.

Since there is no augmenting path, we have $t \notin S$.

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

Let G be a flow with no augmenting paths.

Let S be the set of vertices reachable from s in the residual network.

Let T be the set of vertices not in S .

Since there is no augmenting path, we have

for every edge $(u, v) \in E$ with $u \in S$ and $v \in T$.

Since there is no augmenting path, we have

for every edge $(u, v) \in E$ with $u \in T$ and $v \in S$.

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

1. \Rightarrow 2.

This we already showed.

2. \Rightarrow 3.

If there were an augmenting path, we could improve the flow.
Contradiction.

3. \Rightarrow 1.

- ▶ Let f be a flow with no augmenting paths.
- ▶ Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- ▶ Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$\text{val}(f)$

Augmenting Path Algorithm

$$\text{val}(f) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e)$$

Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e)\end{aligned}$$

Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e) \\ &= \text{cap}(A, V \setminus A)\end{aligned}$$

Augmenting Path Algorithm

$$\begin{aligned}\text{val}(f) &= \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{into}(A)} f(e) \\ &= \sum_{e \in \text{out}(A)} c(e) \\ &= \text{cap}(A, V \setminus A)\end{aligned}$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A .

Analysis

Assumption:

All capacities are integers between 1 and C .

Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.

Analysis

Assumption:

All capacities are integers between 1 and C .

Invariant:

Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral throughout the algorithm.

Lemma 4

The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Theorem 5

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

Lemma 4

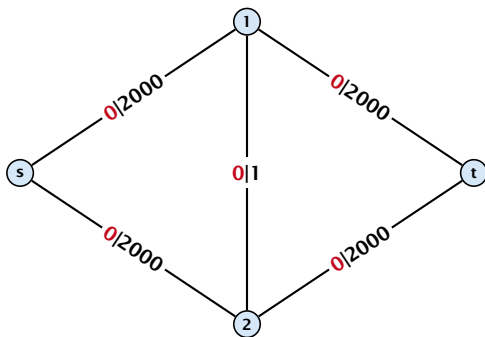
The algorithm terminates in at most $\text{val}(f^*) \leq nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Theorem 5

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

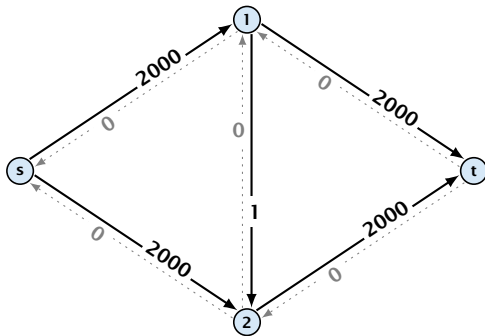
A Bad Input

Problem: The running time may not be polynomial.



A Bad Input

Problem: The running time may not be polynomial.

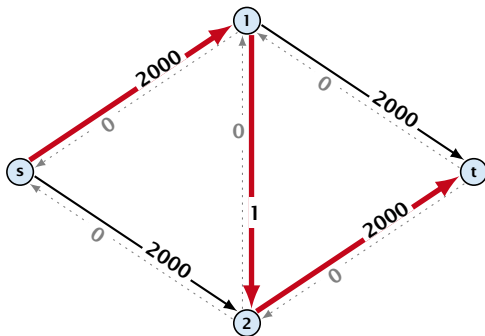


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

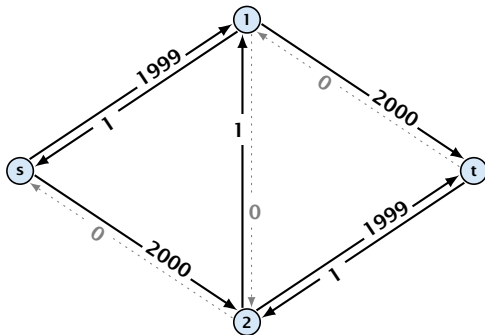


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

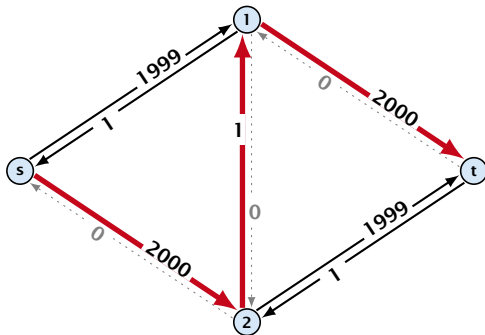


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

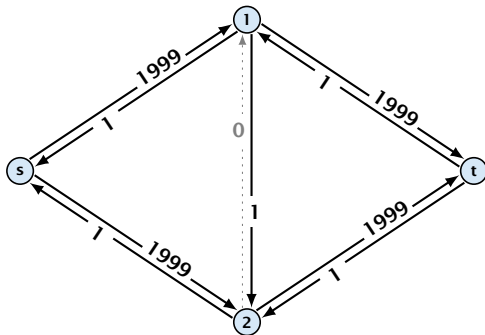


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

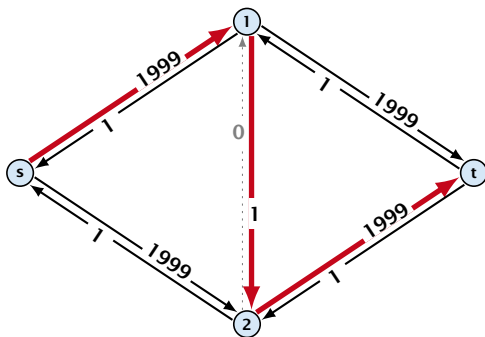


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

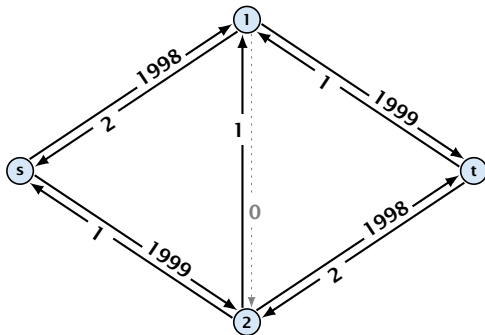


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

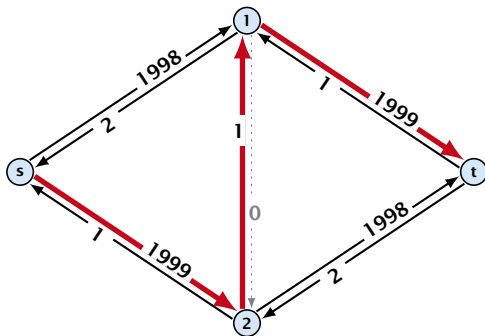


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

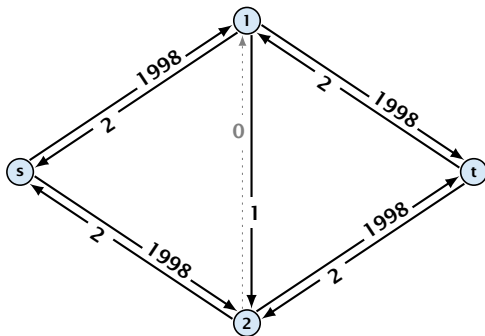


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

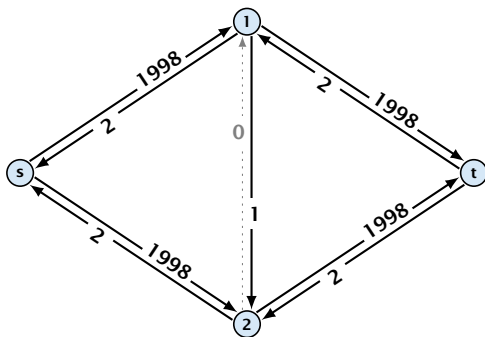


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

A Bad Input

Problem: The running time may not be polynomial.

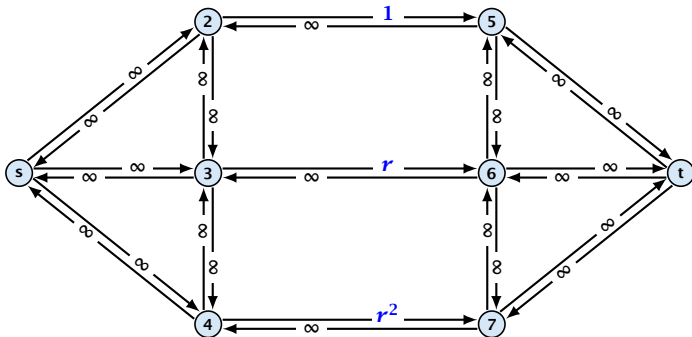


Question:

Can we tweak the algorithm so that the running time is polynomial in the input length?

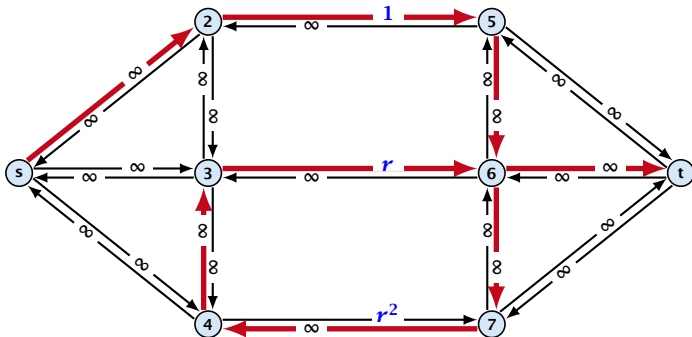
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



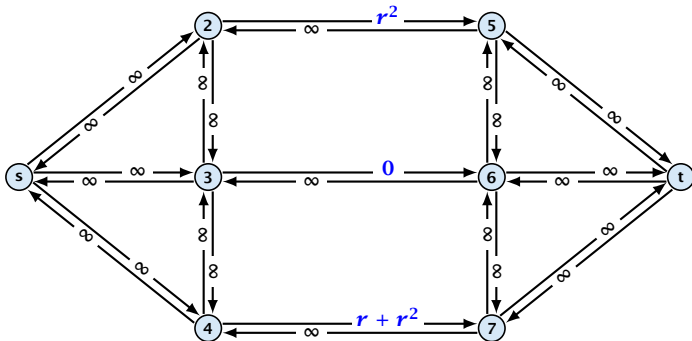
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



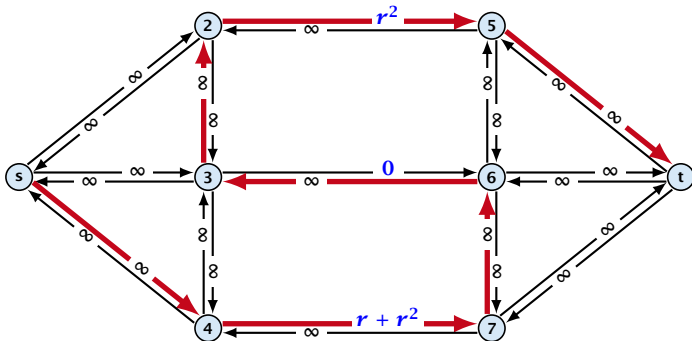
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



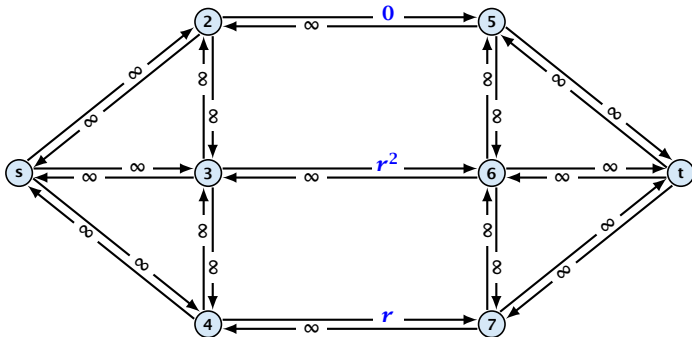
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



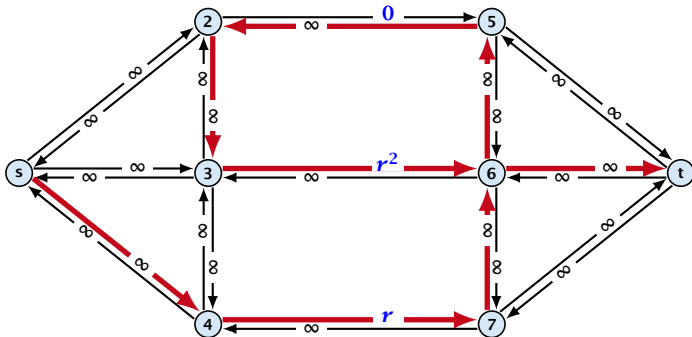
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



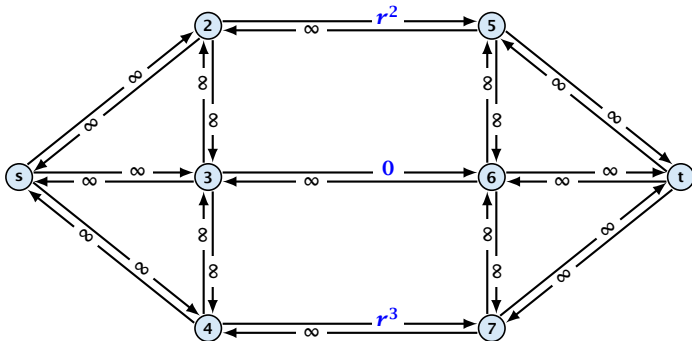
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



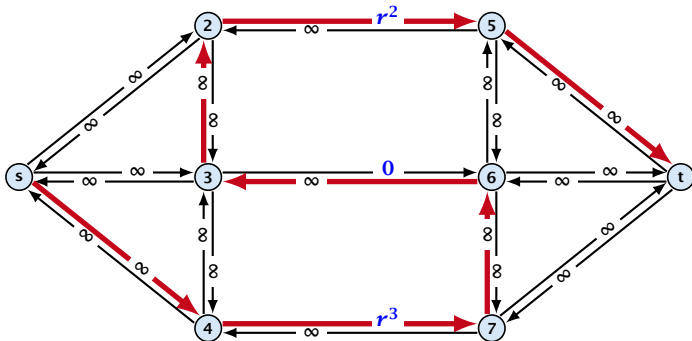
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



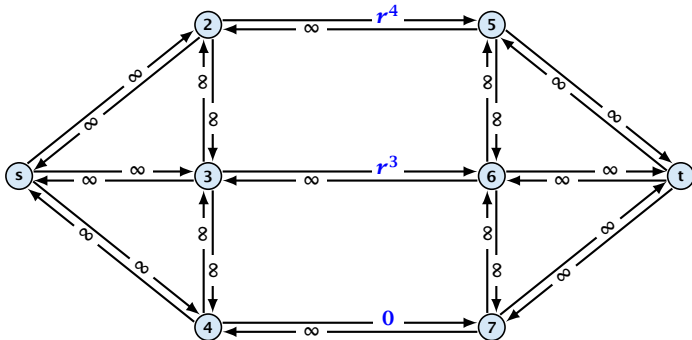
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



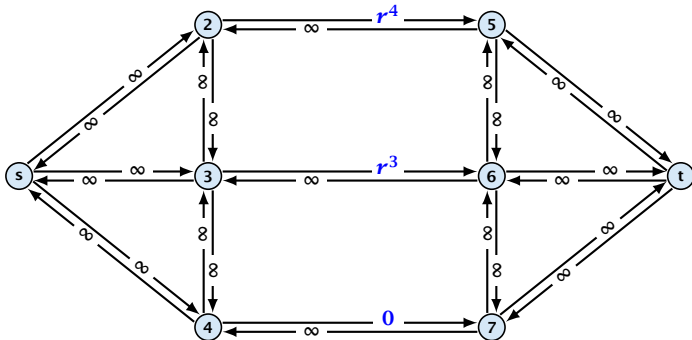
A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



Running time may be infinite!!!

How to choose augmenting paths?

How to choose augmenting paths?

- ▶ We need to find paths efficiently.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Overview: Shortest Augmenting Paths

Lemma 6

The length of the shortest augmenting path never decreases.

Lemma 7

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

Lemma 6

The length of the shortest augmenting path never decreases.

Lemma 7

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

Lemma 6

The length of the shortest augmenting path never decreases.

Lemma 7

After at most $\mathcal{O}(m)$ augmentations, the length of the shortest augmenting path strictly increases.

Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 8

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

We can find the shortest augmenting paths in time $\mathcal{O}(m)$.

Why?

At most n augmentations for paths of exactly n edges.



Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 8

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.



Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 8

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.



Overview: Shortest Augmenting Paths

These two lemmas give the following theorem:

Theorem 8

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. This gives a running time of $\mathcal{O}(m^2n)$.

Proof.

- ▶ We can find the shortest augmenting paths in time $\mathcal{O}(m)$ via BFS.
- ▶ $\mathcal{O}(m)$ augmentations for paths of exactly $k < n$ edges.



Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f .

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f .

Let L_G denote the **subgraph** of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f .

Let L_G denote the **subgraph** of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

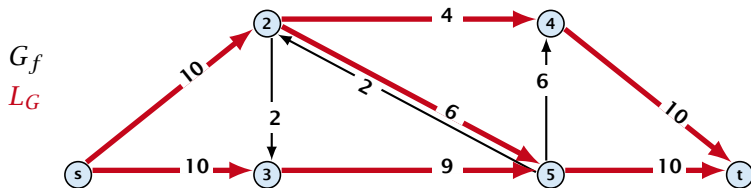
A path P is a shortest s - u path in G_f if it is a an s - u path in L_G .

Shortest Augmenting Paths

Define the level $\ell(v)$ of a node as the length of the shortest s - v path in G_f .

Let L_G denote the **subgraph** of the residual graph G_f that contains only those edges (u, v) with $\ell(v) = \ell(u) + 1$.

A path P is a shortest s - t path in G_f if it is a an s - t path in L_G .



In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

Shortest Augmenting Path

First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t .

Shortest Augmenting Path

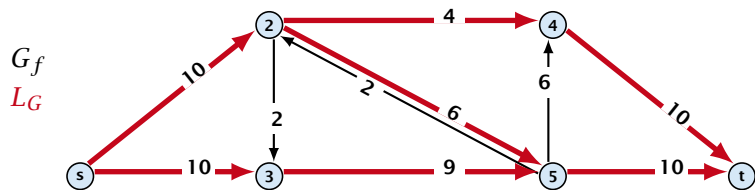
First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t .



Shortest Augmenting Path

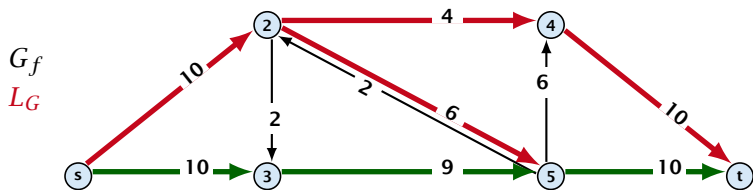
First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t .



Shortest Augmenting Path

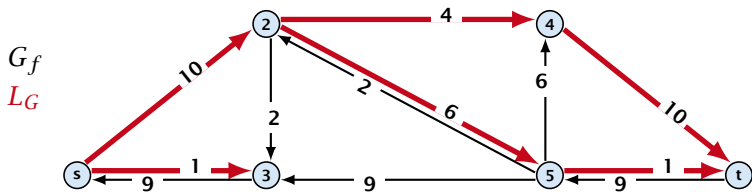
First Lemma:

The length of the shortest augmenting path never decreases.

After an augmentation G_f changes as follows:

- ▶ Bottleneck edges on the chosen path are deleted.
- ▶ Back edges are added to all edges that don't have back edges so far.

These changes cannot decrease the distance between s and t .



Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

An s - t path in G_f that uses edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

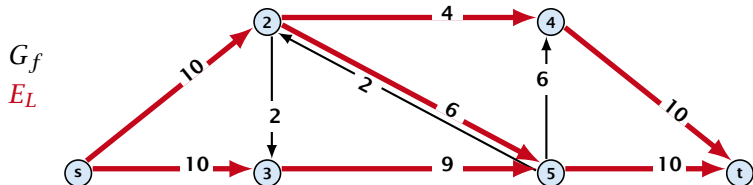
Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

An s - t path in G_f that uses edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

In each augmentation one edge is deleted from E_L .



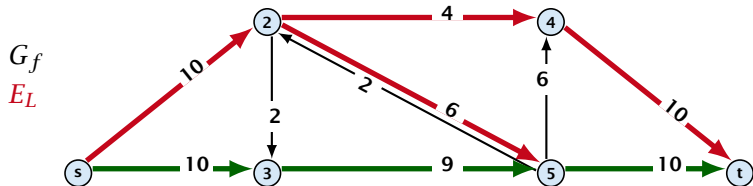
Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

An s - t path in G_f that uses edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

In each augmentation one edge is deleted from E_L .



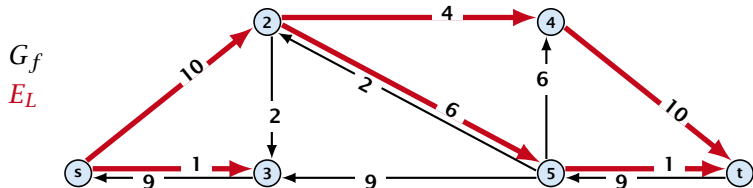
Shortest Augmenting Path

Second Lemma: After at most m augmentations the length of the shortest augmenting path strictly increases.

Let E_L denote the set of edges in graph L_G at the beginning of a round when the distance between s and t is k .

An s - t path in G_f that uses edges not in E_L has length larger than k , even when considering edges added to G_f during the round.

In each augmentation one edge is deleted from E_L .



Shortest Augmenting Paths

Theorem 9

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 10 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

Theorem 9

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 10 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

Theorem 9

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 10 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

Theorem 9

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 10 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

Note:

There always exists a set of m augmentations that gives a maximum flow (why?).

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

When sticking to shortest augmenting paths we cannot improve (asymptotically) on the number of augmentations.

However, we can improve the running time to $\mathcal{O}(mn^2)$ by improving the running time for finding an augmenting path (currently we assume $\mathcal{O}(m)$ per augmentation for this).

Shortest Augmenting Paths

We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s - t path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an s - t path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

Shortest Augmenting Paths

We maintain a subset E_L of the edges of G_f with the guarantee that a shortest $s-t$ path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an $s-t$ path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

Shortest Augmenting Paths

We maintain a subset E_L of the edges of G_f with the guarantee that a shortest $s-t$ path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an $s-t$ path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

Shortest Augmenting Paths

We maintain a subset E_L of the edges of G_f with the guarantee that a shortest $s-t$ path using only edges from E_L is a shortest augmenting path.

With each augmentation some edges are deleted from E_L .

When E_L does not contain an $s-t$ path anymore the distance between s and t strictly increases.

Note that E_L is not the set of edges of the level graph but a subset of level-graph edges.

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a DFS search to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a DFS search to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a **DFS search** to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a **DFS search** to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Suppose that the initial distance between s and t in G_f is k .

E_L is initialized as the level graph L_G .

Perform a **DFS search** to find a path from s to t using edges from E_L .

Either you find t after at most n steps, or you end at a node v that does not have any outgoing edges.

You can delete incoming edges of v from E_L .

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation **during** a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

There are at most n phases. Hence, total cost is $\mathcal{O}(mn^2)$.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

How to choose augmenting paths?

- ▶ We need to find paths efficiently.
- ▶ We want to guarantee a small number of iterations.

Several possibilities:

- ▶ Choose path with maximum bottleneck capacity.
- ▶ Choose path with sufficiently large bottleneck capacity.
- ▶ Choose the shortest augmenting path.

Capacity Scaling

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ .

Capacity Scaling

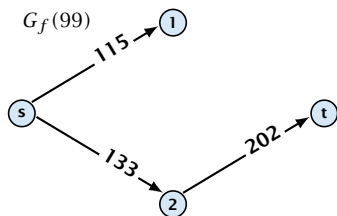
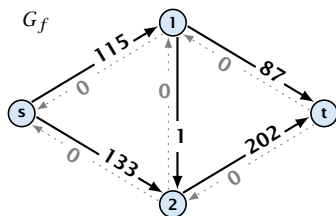
Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ .
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ .

Capacity Scaling

Intuition:

- ▶ Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- ▶ Don't worry about finding the exact bottleneck.
- ▶ Maintain scaling parameter Δ .
- ▶ $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ .



Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

```
1: foreach  $e \in E$  do  $f_e \leftarrow 0$ ;  
2:  $\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$   
3: while  $\Delta \geq 1$  do  
4:    $G_f(\Delta) \leftarrow \Delta$ -residual graph  
5:   while there is augmenting path  $P$  in  $G_f(\Delta)$  do  
6:      $f \leftarrow \text{augment}(f, c, P)$   
7:      $\text{update}(G_f(\Delta))$   
8:    $\Delta \leftarrow \Delta/2$   
9: return  $f$ 
```

Capacity Scaling

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have $G_f(1) = G_f$

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have $G_f(1) = G_f$
- ▶ therefore after the last phase there are no augmenting paths anymore

Capacity Scaling

Assumption:

All capacities are integers between 1 and C .

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- ▶ because of integrality we have $G_f(1) = G_f$
- ▶ therefore after the last phase there are no augmenting paths anymore
- ▶ this means we have a maximum flow.

Capacity Scaling

Capacity Scaling

Lemma 11

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Capacity Scaling

Lemma 11

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Lemma 12

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

Proof: less obvious, but simple:

Capacity Scaling

Lemma 11

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Lemma 12

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

Proof: less obvious, but simple:

- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.

Capacity Scaling

Lemma 11

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Lemma 12

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

Proof: less obvious, but simple:

- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.

Capacity Scaling

Lemma 11

There are $\lceil \log C \rceil$ iterations over Δ .

Proof: obvious.

Lemma 12

Let f be the flow at the end of a Δ -phase. Then the maximum flow is smaller than $\text{val}(f) + m\Delta$.

Proof: less obvious, but simple:

- ▶ There must exist an s - t cut in $G_f(\Delta)$ of zero capacity.
- ▶ In G_f this cut can have capacity at most $m\Delta$.
- ▶ This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Capacity Scaling

Lemma 13

There are at most $2m$ augmentations per scaling-phase.

Capacity Scaling

Lemma 13

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.

Capacity Scaling

Lemma 13

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$

Capacity Scaling

Lemma 13

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ Each augmentation increases flow by Δ .

Capacity Scaling

Lemma 13

There are at most $2m$ augmentations per scaling-phase.

Proof:

- ▶ Let f be the flow at the end of the previous phase.
- ▶ $\text{val}(f^*) \leq \text{val}(f) + 2m\Delta$
- ▶ Each augmentation increases flow by Δ .

Theorem 14

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.