

## 6.1 Guessing+Induction

First we need to get rid of the  $\mathcal{O}$ -notation in our recurrence:

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One way of solving such a recurrence is to **guess** a solution, and check that it is correct by plugging it in.

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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Hence, statement is **true** if we choose  $d \geq c$ .

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Note that we can do this as for constant-sized inputs the running time is always some constant ( $b$  in the above case).

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for a suitable choice of  $d$ .