6.3 The Characteristic Polynomial

Consider the recurrence relation:

```
c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)
```

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- T(n) only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of T[n]'s.
- If f(n) = 0 then the recurrence relation becomes a linear, homogenous recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

© Harald Räcke	
OHarald Räcko	

66

The Homogenous Case

The solution space

 $S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha \mathcal{T}_1 + \beta \mathcal{T}_2 \in S$, for arbitrary constants α, β .

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2\cdot\lambda^{n-2} + \cdots + c_k\cdot\lambda^{n-k} = 0$$

for all $n \ge k$.

```
Marald Räcke
```

6.3 The Characteristic Polynomial

68

6.3 The Characteristic Polynomial

Observations:

- The solution T[1], T[2], T[3],... is completely determined by a set of boundary conditions that specify values for T[1],...,T[k].
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
- First consider the homogenous case.

© Harald Räcke

6.3 The Characteristic Polynomial

The Homogenous Case

Dividing by λ^{n-k} gives that all these constraints are identical to

$$\underbrace{c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k}_{\text{characteristic polynomial } P[\lambda]} = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

$$\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$$

is a solution for arbitrary values α_i .



67

The Homogenous Case

Lemma 1

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

 $\alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \cdots + \alpha_k\lambda_k^n$.

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

CHarald Räcke	6.3 The Characteristic Polynomial	
🛛 🕒 🖯 🖾 © Harald Räcke		70

The Homogenous Case

Proof (cont.).

Suppose I am given boundary conditions T[i] and I want to see whether I can choose the $\alpha'_i s$ such that these conditions are met:

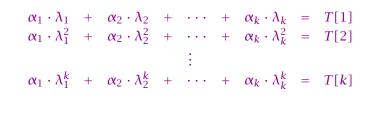
$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ & \vdots & & \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_k^k \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} T[1] \\ T[2] \\ \vdots \\ T[k] \end{pmatrix}$$

We show that the column vectors are linearly independent. Then the above equation has a solution.

The Homogenous Case

Proof (cont.).

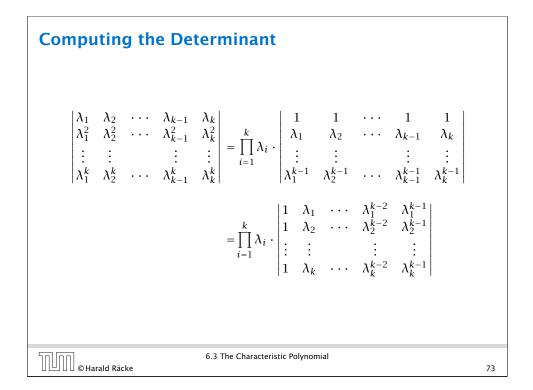
Suppose I am given boundary conditions T[i] and I want to see whether I can choose the $\alpha'_i s$ such that these conditions are met:



Marald Räcke

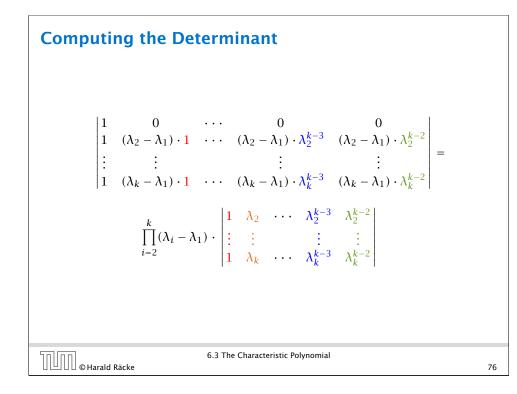
6.3 The Characteristic Polynomial

71

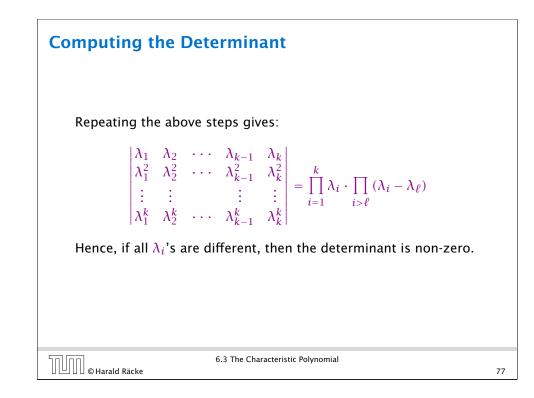


© Harald Räcke

Computing the Determinant		
$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix} =$		
$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$		
6.3 The Characteristic Polynomial		



$\begin{array}{c} \left| \begin{array}{c} 1 \quad \lambda_{1} - \lambda_{1} \cdot 1 \quad \cdots \quad \lambda_{1}^{k-2} - \lambda_{1} \cdot \lambda_{1}^{k-3} \quad \lambda_{1}^{k-1} - \lambda_{1} \cdot \lambda_{1}^{k-2} \\ 1 \quad \lambda_{2} - \lambda_{1} \cdot 1 \quad \cdots \quad \lambda_{2}^{k-2} - \lambda_{1} \cdot \lambda_{2}^{k-3} \quad \lambda_{2}^{k-1} - \lambda_{1} \cdot \lambda_{2}^{k-2} \\ \vdots \quad \vdots \quad & \vdots \quad & \vdots \\ 1 \quad \lambda_{k} - \lambda_{1} \cdot 1 \quad \cdots \quad \lambda_{k}^{k-2} - \lambda_{1} \cdot \lambda_{k}^{k-3} \quad \lambda_{k}^{k-1} - \lambda_{1} \cdot \lambda_{k}^{k-2} \\ \end{array} \right| = \\ \left| \begin{array}{c} 1 \quad 0 \quad \cdots \quad 0 \quad 0 \quad 0 \\ 1 \quad (\lambda_{2} - \lambda_{1}) \cdot 1 \quad \cdots \quad (\lambda_{2} - \lambda_{1}) \cdot \lambda_{2}^{k-3} \quad (\lambda_{2} - \lambda_{1}) \cdot \lambda_{2}^{k-2} \\ \vdots \quad \vdots \quad & \vdots \quad & \vdots \\ 1 \quad (\lambda_{k} - \lambda_{1}) \cdot 1 \quad \cdots \quad (\lambda_{k} - \lambda_{1}) \cdot \lambda_{k}^{k-3} \quad (\lambda_{k} - \lambda_{1}) \cdot \lambda_{k}^{k-2} \\ \end{array} \right| \\ \end{array} \right| = \\$



The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

 $P[\lambda] \cdot \lambda^{n-k} = c_0 \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \dots + c_k \lambda^{n-k}$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$. Calculating the derivative gives a polynomial that still has root λ_i .

החוחר	6.3 The Characteristic Polynomial	
UUU© Harald Räcke		78

The Homogeneous Case

Suppose λ_i has multiplicity j. We know that

 $c_0 n \lambda_i^n + c_1 (n-1) \lambda_i^{n-1} + \dots + c_k (n-k) \lambda_i^{n-k} = 0$

(after taking the derivative; multiplying with λ ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue j - 1 times.

```
Hence, n^{\ell}\lambda_i^n is a solution for \ell \in 0, ..., j-1.
```

```
©Harald Räcke
```

6.3 The Characteristic Polynomial

80

This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$



$$c_0 \underbrace{n\lambda_i^n}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

The Homogeneous Case

Lemma 2

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

 $c_0T[n] + c_1T[n-1] + \cdots + c_kT[n-k] = 0$

Let λ_i , i = 1, ..., m be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.



Example: Fibonacci Sequence

T[0] = 0 T[1] = 1T[n] = T[n-1] + T[n-2] for $n \ge 2$

The characteristic polynomial is

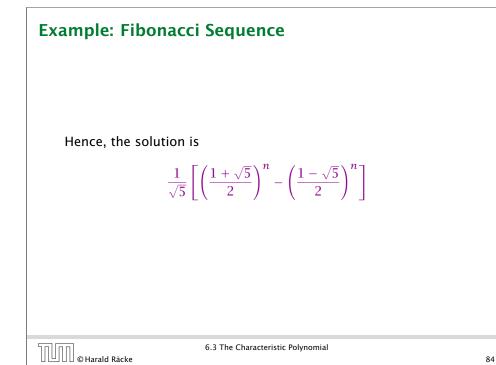
 $\lambda^2 - \lambda - 1$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} \left(1 \pm \sqrt{5} \right)$$

© Harald Räcke

6.3 The Characteristic Polynomial



Example: Fibonacci Sequence

Hence, the solution is of the form

$\alpha \left(\frac{1+\sqrt{5}}{2}\right)^n + \beta$	$B\left(\frac{1-\sqrt{5}}{2}\right)^n$
--	--

T[0] = 0 gives $\alpha + \beta = 0$.

T[1] = 1 gives

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1 \Longrightarrow \alpha-\beta=\frac{2}{\sqrt{5}}$$

© Harald Räcke

82

6.3 The Characteristic Polynomial

83

The Inhomogeneous Case

Consider the recurrence relation:

 $c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

 $T(n) = T_h(n) + T_p(n) ,$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

G Harald Räcke

6.3 The Characteristic Polynomial

The Inhomogeneous Case

Example: Characteristic polynomial:

 $\underbrace{\lambda^2 - 2\lambda + 1}_{(\lambda - 1)^2} = 0$

Then the solution is of the form

$$T[n] = \alpha 1^n + \beta n 1^n = \alpha + \beta n$$

T[0] = 1 gives $\alpha = 1$.

T[1] = 2 gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

الله المعام المحالة (@Harald Räcke

6.3 The Characteristic Polynomial

The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1$$
 $T[0] = 1$

Then,

T[n-1] = T[n-2] + 1 $(n \ge 2)$

Subtracting the first from the second equation gives,

 $T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \ge 2)$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \ge 2)$$

I get a completely determined recurrence if I add T[0] = 1 and T[1] = 2.

Marald Räcke

6.3 The Characteristic Polynomial

87

The Inhomogeneous Case

If f(n) is a polynomial of degree r this method can be applied r + 1 times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2 = T[n-2] + n^2 - 2n + 1$$

Difference:

$$T[n] - T[n-1] = T[n-1] - T[n-2] + 2n - 1$$

T[n] = 2T[n-1] - T[n-2] + 2n - 1

86

T[n] = 2T[n-1] - T[n-2] + 2n - 1

Shift:

$$T[n-1] = 2T[n-2] - T[n-3] + 2(n-1) - 1$$
$$= 2T[n-2] - T[n-3] + 2n - 3$$

Difference:

$$T[n] - T[n-1] = 2T[n-1] - T[n-2] + 2n - 1$$
$$- 2T[n-2] + T[n-3] - 2n + 3$$

$$T[n] = 3T[n-1] - 3T[n-2] + T[n-3] + 2$$

and so on...





